QUADRATIC PROGRAMMING

DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF THE DEGREE OF

Master of Philosophy

in

Operations Research

BY

NEHA GUPTA

UNDER THE SUPERVISION OF

DR. MOHD. ARSHAD

DEPARTMENT OF STATISTICS & OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002 (INDIA)

2011

Dedicated To My Parents Who Give Their Today For My Better Tomorrow.

DEPARTMENT OF
STATISTICS & OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002, INDIA.

Dr. Mohd. Arshad

Date:

Certificate

This is to certify that **Ms. Neha Gupta** has carried out the work reported in the present dissertation entitled **"Quadratic Programming"** under my supervision and her dissertation work is suitable for submission for the degree of *Master of Philosophy* in **Operations Research.**

(Dr. Mohd. Arshad)
Supervisor

Acknowledgements

Firstly, the one above all of us, the omnipresent God, for answering my prayers for giving me the strength to plod on despite my constitution wanting to give up and throw in the towel, thank you so much Dear Lord.

I am heartily thankful to my supervisor, Dr. Mohd. Arshad, Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, whose encouragement, supervision and support from the preliminary to the concluding level enabled me to develop an understanding of the subject.

I would like to express my deepest gratitude to Prof. Abdul Bari, Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, a gracious mentor, for his excellent guidance, caring, patience, and providing me with an excellent atmosphere for doing research despite of his busy schedule.

I am especially indebted to Prof. Mohammad Masood Khalid, Chairman, Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, who made available his support in a number of ways.

It is an honor for me to show my deepest gratitude to my professors for showing me by example and through challenging coursework how to think, teach, and teach others to think. Prof. A. H. Khan, Prof. M. J. Ahsan, Prof. I. A. Khan, Prof. Q. M. Ali, Prof. Athar Ali Khan, Prof. M. Yaqub, Mr. S. S. Hasan, Dr. R. U. Khan, Dr. Haseeb Athar, Dr. Shakeel Javed and Dr. Kamalullah.

I would also like to express my thanks to the non-teaching staff for their assistance especially Mr. Firoz, Mr. Rizwan, Mr. Zulfikar, Mr. Jahangir and Mr. Shabbir in the seminar library.

I would like to thank Sanam Haseen and Shubhra Singh, who as good friends, was always willing to help and give their best suggestions. It would have been a lonely lab without them.

I would like to show my gratitude to my senior colleagues for sharing their enthusiasm and comments on my work especially Mr Yashpal Singh Raghav, Mr. Rahul Varshney, Mr. Devendra Kumar, Ms. Yojna Yadav, Mr M. Imtiyaz, Mrs. Sanghmitra Sharma, Ms. Farha Naz, Ms.Ummatul Fatima, Ms Sana Iftekar, Ms Anamika Kulshrestha, Ms Zubdah-e-Noor, Mr Irfan Ali, Mr. Shamsher Ali Khan and Mr Shafiullah.

Words are inadequate in offering my thanks to affectionate friends especially Ms. Sadia Anwar, Ms. Yasmin Khan, Mr. Tanwir Akhtar, Mr. Ashfaq Ahmed Khan, Mr. Safayat Kareem, Mr. Mustafa Kamal and Mr. Zubair Akhter.

Where would I be without my family? My parents deserve special mention for their inseparable support and prayers. My Father in the first place is the person who put the fundament my learning character, showing me the joy of intellectual pursuit ever since I was a child. My Mother is the one who sincerely raised me with her caring and gently love. I would also like to thank my elder sister and elder brother for being supportive and caring.

I wish to thank the Aligarh Muslim University for providing me all the facilities to do my research and UGC for providing me Non-NET UGC Scholarship.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of my dissertation.

Neha Gupta
(Neha Gupta)

Contents

Preface

This dissertation entitled '*Quadratic Programming*' is submitted to Aligarh Muslim University, Aligarh, for the partial fulfillment of the degree of *M.Phil.* In this manuscript an attempt has been made to present a survey of the available literature on quadratic programming.

The dissertation consists of *Five* chapters. The *First* chapter is devoted to the introduction of History of O.R., Optimization problems, Mathematical programming problem, Linear programming problem, Non-Linear programming problem, Quadratic programming problems, Quadratic forms and Duality in Quadratic programming problem.

In chapter *Two,* various methods for solving different types of quadratic programming are discussed. Quadratic Programming is a mathematical programming problem in which the objective function is a sum of linear and quadratic form. For solving Quadratic programming Problem Wolfe's and Beale's method are illustrated. For the solution of Concave Quadratic Programming Problem Tui's cut is also explained in detail.

Chapter *Three* presents some recent methods for solving all integer and mixed integer quadratic programming problems.

In chapter *Four* , some situations in Economics, Industry and Sample surveys have been formulated as the problem of quadratic problem which shows the importance and applications of the topic.

And last but not the least in chapter *Five* a brief introduction of Bilevel programming problems is given and two algorithms for Bilevel Programming Problems are discussed viz. Vertex Enumeration Approach and Kuhn Tucker Approach. Also a method for solving Mixed Integer Concave Quadratic Bilevel programming problem is discussed.

Fairly comprehensive references of various publications referred in this dissertation have been given at the end of this manuscript. The references are arranged alphabetically according to the author's name.

Chapter- One

INTRODUCTION

Background and Historical Sketch of O.R.

Since the beginning of the history of mankind, man has been confronted with the problem of deciding a course of action that would be the best for him under the circumstances. This process of making optional judgment according to various criteria is known as the science of decision making. Unfortunately, there was no scientific method of solution for such an important class of problems until very recently. It is only in 1930's that a systematic approach to the decision problem started developing, mainly due to the advent of the 'New-Deal' in the United States and similar attempts in other parts of the world to curve the great economic depression prevailing throughout the world during this period. As a result during the 1940's, a new science began to emerge out.

About the same time, during world war II, the military management in the United Kingdom called upon a group of scientists from different disciplines to use their scientific knowledge for providing assistance to several strategic and tactical war problems. The encouraging results advised by the British scientists soon motivated the military management of the U.S.A. to start on similar activities.

The methodology applied by these scientists to achieve their objectives was named as O.R. because they were dealing with research on military operations.

Operational research, popularly known as O.R., is a recent addition to a long list of scientific tools which provide a new outlook to many conventional management problems. Operational research adds greater sophistication towards solving management problems. It seeks the determination of best (optimum) course of action of a decision problem under the limiting factor of limited resources. Accordingly O.R. has become a versatile tool in the field of management and its potential for future use is very substantial. O.R. proves an effective scientific technique to solve such decision-making problems of modern business and industry.

Optimization Problem

A key word associated with O.R. is 'optimization'. Optimization means determining the best course of action amongst the different alternatives available in a decision-making problem. It can be regarded as a process of finding the optimal value (the greatest or the smallest as the case may be) of a function (usually called the objective function) under a given set of circumstances (often called 'constraints'). Optimization can, thus, be viewed as a decision-making process or more specifically as one of the major quantitative tools in the network of decision-making, in which decisions have to be taken which optimize one or more of the specified objectives under the prescribed set of constraints.

Optimization problems arise in almost every sphere of human activity. These occur in almost every engineering discipline such as Civil, Mechanical, Electrical, Telecommunication, Chemical and Biochemical, Engineering Design and Manufacturing systems etc. These also occur in Business Administration, Management and other Economic and Industry related fields. In fact, the newly developed optimization techniques are now being applied in every sphere of human activity where decisions have to be taken in some complex situation which can be represented by a mathematical model.

Solving an Optimization problem:

Solution of a real life optimization problem usually involves three phases: i) modeling phase, ii) solution of the mathematical model, and iii) validation of the results and their implementation. Out of these three, the first phase namely 'modeling phase' is the most vital one. An incorrect model yields an incorrect solution. However, the other two phases are also equally important as these provide the basis for obtaining the optimal solution and its implementation in the real life situation.

In a majority of the cases, the real life optimization problem is available in descriptive form in words. It has to be transformed into a mathematical model in which one or more of the available techniques of optimization can be applied. In earlier days in view of the limited availability of the computational facilities, the trend was to introduce approximations and assumptions in the model, so that it could be conveniently solved using some well-known techniques of optimization. However, the solution of this modified and simplified model often did not meet the specifications of the actual end user. This was one of the main reasons why initially the practical users were not so enthusiastic in using these methods. However, with the easy availability of fast computing facilities in the form of personal computers, and at the same time the development of more robust and efficient computational techniques of optimization, the scenario has now been changed. Solution of more realistic and complex problems can now be obtained in more or less their original form and in a relatively much shorter time span.

There can be variety of mathematical models of real life optimization problems which are discussed in further sections.

Mathematical Programming

Mathematical programming is concerned with finding optimal solutions to the problems of decision making under limited resources to meet the desired objectives.

The mathematical programming problem (MPP) can be formulated as:

$$\text{maximize (minimize)}\ z = f(\underline{x}),$$

$$\textit{subject to the constraints}\ g_i[\leq, =, \geq]b_i\ , i = 1, \dots, m$$

$$\textit{and the non}-\textit{negativity resrictions}\ \underline{x} \geq 0,$$

where $\underline{x}= (x_1, \dots, x_n)$ is an n-component vector of variables, $f(\underline{x})$ and $g_i(\underline{x})$ are functions of n-variables $(x_1, \dots, x_n)$ and b_i are known constants. Furthermore one and only one of the signs $\leq$,$=$ and $\geq$ holds for each constraint.

Depending upon the nature of the objective function $f(\underline{x})$, the functions $g_i(\underline{x})$ in the constraints and other restrictions on the variable vector x the MPP may be classified under different headings. Although no single technique has been found to be universally applicable for almost all classes. Some important classes are listed below:

1. Linear programming
2. Non-linear programming
3. Quadratic programming
4. Dynamic programming
5. Integer programming

6. Stochastic programming
7. Goal programming
8. Parametric programming
9. Chance constrained programming
10. Geometric programming
11. Separable programming

[Note that all of these classes are not mutually exclusive]

For the purpose of this dissertation in the succeeding chapters we will study the quadratic programming problem (QPP) in detail.

Linear Programming Problem

Linear programming was developed in 1947 by George B. Dantzig, Marshall Wood, and their associates, as a tool for finding optimal solutions to military planning problems for the United States Air Force. The early applications were primarily limited to problems involving military operations, such as military logistics problems, military transportation problems, procurement problems, and other related fields. In addition, linear programming was applied to inter-industry economic problems.

The uses of linear programming range from the government sector to agricultural, business and industrial sectors. Its uses can also be found in economic theory, dietetics, industrial engineering, and applied mathematics.

Linear programming is a mathematical programming technique most closely associated with operations research and management science. In business, linear programming is used for finding the optimal uses of the firm's limited resources. A linear programming problem is often referred to as an allocation problem because it deals with allocation of resources to alternative uses.

Linear programming involves the formulation and solution of a class of business problems by the optimization of a linear mathematical function subject to linear inequalities. Here the term linear has a specific meaning.

The mathematical formulation of linear programming problem for n decision variables and m side constraints is as follows:

$$\max F = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{aligned} \textit{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \end{aligned}$$

$$\textit{and} \quad x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

where $a_{ij}, b_i, c_j = \textit{given constants}$

$x_j = \textit{decision variables}$

m=*no. of side constraints*

n= *no. of decision variables.*

The above problem is formulated in a more general form as:

$$\max F = \sum_{j=1}^{n} c_j x_j$$

$$\textit{subject to} \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (\textit{for } i = 1,2, \ldots, m)$$

$$and \;\; x_j \geq 0 \quad (for\, j = 1,2,\dots,n)$$

In a similar way, the minimization problem is stated in the form:

$$\min F = \sum_{j=1}^{n} c_j\, x_j$$

$$subject\ to \sum_{j=1}^{n} a_{ij}\, x_j \geq b_i \quad (for\ i = 1,2,\dots,m)$$

$$and \;\; x_j \geq 0 \quad (for\, j = 1,2,\dots,n)$$

The basic difference between the maximization and the minimization problems in linear programming is found in the signs of the inequalities of the side constraints. The side constraints are expressed by "≤" sign in maximization problem; where as those of the minimization problem are expressed by the "≥" sign.

Non-Linear Programming Problem

In linear programming it was assumed that there must exist a linear relationship among all decision variables. However, in many real life problems the assumption of linearity may not exist. For example, the sales prices or sales quantities may decrease as sales volume increases production cost may increase or decrease with changes in certain factors such as marginal productivity of productive factors. As a result, the objective function and/or one or more of the constraints will have non-linear relationships among decision variables. Non-linearity can also arise when any of the cost or profit coefficients in a linear programming model is a random variable.

Like Linear Programming, Non-Linear programming is a mathematical technique for determining the optimal solutions to many business problems. In a non-linear programming problem, either the objective function is non-linear, or one or more constraints have non-linear relationship or both.

Interest in nonlinear programming problems developed simultaneously with the growing interest in linear programming. In the absence of general algorithms for NLPP, it lies near at hand to explore the possibilities of approximate solution by linearization. The nonlinear functions of a mathematical programming problem were replaced by piecewise linear functions, these approximations may be expressed in such a way that the whole problem is turned into linear programming.

Kuhn and Tucker(1951) published an important paper "Non-linear programming", dealing with necessary and sufficient conditions for optimal solutions to programming problems, which laid the foundations for a great deal of later work in non-linear programming.

A mathematical programming problem in which all the involved functions are not linear is called a nonlinear programming problem (NLPP). The mathematical model of an NLPP may be given as:

$$\text{maximize(or minimize)}\, f(x_1, x_2, \dots, x_n)$$

$$s.t.\ \ g_i(x_1, x_2, \dots, x_n)\{\leq or = \geq\}b_i\ \ i = 1, \dots, m$$

$$and\ \ x_j \geq 0; \quad j = 1,2, \dots, n$$

where $f(x_1, x_2, \dots, x_n)$ and $g_i(x_1, x_2, \dots, x_n)$ are real valued function of n decision variables and at least one of these is non-linear. Several methods have been developed for solving non-linear programming problems.

Quadratic Forms

Definition: The function $Q(\underline{x})$ of n-variables $\underline{x} = (x_1, x_2, \ldots, x_n)$ is said to be quadratic form if

$$Q(\underline{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}\, x_i x_j, i, j = 1, \ldots, n$$

In matrix notations $Q(\underline{x}) = \underline{x}'D\underline{x}$, where $D = ((d_{ij}))$, without loss of generality we can assume that D is symmetric.

Types of Quadratic forms: A quadratic form $Q(\underline{x}) = \underline{x}'D\underline{x}$ is said to be:

- Positive definite if $\underline{x}'D\underline{x} > 0$ for all $\underline{x} \neq 0$.
- Positive semi definite if $\underline{x}'D\underline{x} \geq 0$ for all x and there exists at least one $x \neq 0$ such that $\underline{x}'D\underline{x} = 0$.
- Negative definite if $-\underline{x}'D\underline{x}$ is positive definite.
- Negative semi definite if $-\underline{x}'D\underline{x}$ is positive semi definite.
- Indefinite if it does not fall in any of the above four categories.

Some properties of quadratic forms:

- A positive semi definite quadratic form is a convex function.
- The definiteness of a quadratic form is invariant under non-singular linear transformation.
- Every quadratic form can be reduced to a form containing square terms only by a non-singular transformation.
- If $\underline{x}'D\underline{x}$ is positive definite there exists a non-singular transformation.

  $$Y = XX_{\beta} \text{ such that } x'Dx \rightarrow Y'Y = y_1^2 + y_2^2 + \cdots + y_n^2.$$

- The necessary and sufficient condition that a real quadratic form $\underline{x}'D\underline{x}$ is positive definite is that $d_i > 0$ for $i = 1,2, \ldots, n$.

where $$d_i = \begin{vmatrix} d_{11} & \cdots & d_{22} \\ \vdots & & \vdots \\ d_{i1} & \dots & d_{ii} \end{vmatrix}$$

Quadratic Programming Problem

In this section we give the definition of quadratic program and discuss the nature of the coefficients and that of the functions involved so that quadratic program becomes feasible and finite solution exists.

Quadratic programming (QP) deals with a special class of mathematical programs in which a quadratic function of the decision variables is required to be optimized (i.e., either minimized or maximized) subject to linear equality and/or inequality constraints. Let $\underline{x} = (x_1, \dots, x_n)^T$ denote the column vector of decision variables. In mathematical programming, it is standard practice to handle a problem requiring the maximization of a function $f(\underline{x})$ subject to some constraints by minimizing $-f(\underline{x})$ subject to the same constraints. Both problems have the same set of optimum solutions. Because of this, we restrict our discussion to minimization problems.

A quadratic function is the simplest nonlinear function, and hence they have always served as model functions for approximating general nonlinear functions by local models (through Taylor series and other such approximations). Hence, quadratic programming models serve as a bridge between linear programming and nonlinear programming models.

The general quadratic program can be written as

$$\text{Minimize} f(\underline{x}) = \underline{c}\ \underline{x} + \frac{1}{2}\underline{x}^T D\underline{x}$$

$$subject\ to\ A\underline{x} \leq \underline{b}\ and\ \underline{x} \geq 0$$

where c is an n-dimensional row vector describing the coefficients of the linear terms in the objective function, and D is an $(n \times n)$ symmetric matrix describing the coefficients of the quadratic terms. If a constant term exists it is dropped from the model. As in linear programming, the decision variables are denoted by the n-dimensional column vector x, and the constraints are defined by an $(m \times n)$ A matrix and an m-dimensional column vector b of right-hand-side coefficients. We assume that a feasible solution exists and that the constraint region is bounded.

When the objective function $f(\underline{x})$ is strictly convex for all feasible points the problem has a unique local minimum which is also the global minimum. A sufficient condition to guarantee strict convexity for D to be positive definite.

In quadratic programming problem the structural relation among the variables is assumed to be known. Our aim is to determine the optimal policies subject to the known structural restrictions and the condition of nonnegativity of the variables. The real problem situations do not allow the variables to have negative values. As implied, the optimum solution point either maximizes or minimizes some linear or nonlinear combination of the decision variables.

Although quadratic programming problems call for the determination of a global optimum, numerical techniques will, in general, lead to a local optimum. On the more, it is not possible to determine if a local optimum is really a global optimum. Even if it could be done, quadratic programming procedures have no way of proceeding from a local optimum to a global optimum.

Fortunately, mathematical tools have been developed to establish the coincidence of the local and global optima and a number of computational procedures have been framed for finding a global optimum for quadratic programming problems for those cases where it is known that any local optimum is also a global optimum.

Karush-Kuhn-Tucker Conditions

We now specialize the general first-order necessary conditions to the quadratic program. These conditions are sufficient for a global minimum when Q is positive definite; otherwise, the most we can say is that they are necessary.

Excluding the nonnegativity conditions, the Lagrangian function for the quadratic program is

$$L\left(\underline{x},\underline{\mu}\right) = \underline{cx} + \tfrac{1}{2}\underline{x}^T Q\underline{x} + \underline{\mu}(A\underline{x} - \underline{b}),$$

where μ is an m-dimensional row vector. The Karush-Kuhn-Tucker conditions for a local minimum are given as follows.

$$\frac{\partial L}{\partial x_j} \geq 0, j = 1, \dots, n \qquad \underline{c} + Q\underline{x}^T + \underline{\mu}A \geq 0 \qquad (a)$$

$$\frac{\partial L}{\partial \mu_i} \leq 0, i = 1, \dots, m \qquad A\underline{x} - \underline{b} \leq 0 \qquad (b)$$

$$x_j \frac{\partial L}{\partial x_j} = 0, j = 1, \dots, n \qquad \underline{x}^T\left(\underline{c}^T + Q\underline{x} + A^T\underline{\mu}\right) = 0 \qquad (c)$$

$$\mu_i g_i(x) = 0, i = 1, \dots, m \qquad \underline{\mu}\left(A\underline{x} - \underline{b}\right) = 0 \qquad (d)$$

$$x_j \geq 0, j = 1, \dots, n \qquad \underline{x} \geq 0 \qquad (e)$$

$$\mu_i \geq 0, i = 1, \dots, m \qquad \underline{\mu} \geq 0 \qquad (f)$$

To put (a)-(f) into a more manageable form we introduce nonnegative surplus variables $y \in \Re^{n}$ to the inequalities in (a) and nonnegative slack variables $v \in \Re^{m}$ to the inequalities in (b) to obtain the equations

$$\underline{c}^T + Q\underline{x} + A^T\underline{\mu}^T - \underline{y} = 0 \text{ and } A\underline{x} - \underline{b} + \underline{v} = 0.$$

The KKT conditions can now be written with the constants moved to the right-hand side.

$$Q\underline{x} + A^T\underline{\mu}^T - \underline{y} = -\underline{c}^T \qquad (a)$$

$$A\underline{x} + \underline{v} = \underline{b} \qquad (b)$$

$$\underline{x} \geq 0, \underline{\mu} \geq 0, \underline{y} \geq 0, \underline{v} \geq 0 \qquad (c)$$

$$\underline{y}^T\underline{x} = 0, \underline{\mu}\underline{v} = 0 \qquad (d)$$

The first two expressions are linear equalities, the third restricts all the variables to be nonnegative, and the fourth prescribes complementary slackness.

The simplex algorithm can be used to solve (a)-(d) by treating the complementary slackness conditions (d) implicitly with a restricted basis entry rule. The procedure for setting up linear programming model follows.

Let the structural constraints be Eqs. (a) and (b) defined by the KKT conditions.

If any of the right-hand-side values are negative, multiply the corresponding equation by -1.

- Add an artificial variable to each equation.
- Let the objective function be the sum of the artificial variables.
- Put the resultant problem into simplex form.

The goal is to find the solution to the linear program that minimizes the sum of the artificial variables with the additional requirement that the complementarily slackness conditions be satisfied at each iteration. If the sum is zero, the solution will satisfy (a)-(d). To accommodate (d), the rule for selecting the entering variable must be modified with the following relationships in mind.

x_j and y_j are complementary for $j = 1, \dots, n$

μ_i and v_i are complementary for $i = 1, \dots, m$

The entering variable will be the one whose reduced cost is most negative provided that its complementary variable is not in the basis or would leave the basis on the same iteration. At the conclusion of the algorithm, the vector x defines the optimal solution and the vector μ defines the optimal dual variables.

This approach has been shown to work well when the objective function is positive definite, and requires computational effort comparable to a linear programming problem with $m + n$ constraints, where m is the number of constraints and n is the number of variables in the QP. Positive semi-definite forms of the objective function, though, can present computational difficulties. Van De Panne (1975) presents an extensive discussion of the conditions that will yield a global optimum even when $f(\underline{x})$ is not positive definite. The simplest practical approach to overcome any difficulties caused by semi-definiteness is to add a small constant to each of the diagonal elements of Q in such a way that the modified Q matrix becomes positive definite. Although the resultant solution will not be exact, the difference will be insignificant if the alterations are kept small.

Classification of Quadratic Programs

Quadratic Programs can be classified into the following types:

Unconstrained quadratic minimization problem is one that requires the minimization of a quadratic function $Q(\underline{x})$ over the whole space $\Re^n$ with no constraints.

Equality constrained quadratic minimization problem is one that requires the minimization of a quadratic function $Q(\underline{x})$ subject to linear equality constraints on the variables, $A\underline{x} = \underline{b}$. These equations can be used to eliminate some variables by expressing them in terms of the others, and thereby transform the problem into an unconstrained one in the remaining variables. Thus, these problems are mathematically equivalent to (and can be solved by techniques similar to those of) unconstrained quadratic minimization problems.

Inequality constrained quadratic minimization problem is one that requires the minimization of a quadratic function $Q(\underline{x})$ subject to linear inequality constraints $B\underline{x} \geq \underline{d}$, and possibly bounds on individual variables $l \leq x \leq u$, and may be some equality constraints $A\underline{x} = \underline{b}$.

Bound constrained quadratic minimization problem is one that requires the minimization of a quadratic function $Q(\underline{x})$ subject only to bounds (lower and/or upper) on the variables.

Convex Quadratic Programming

This section gives the definition of convex quadratic program and introduces concepts of convex function, convex set of feasible solutions and discusses solvability of the convex quadratic program.

Convex quadratic programming is an important class of convex programs in which the objective function is quadratic and convex and the constraints are linear. The objective function may be a sum of a linear form and a convex quadratic form and hence is also convex. The standard formulation is the following:

$$\begin{aligned} &\text{minimize}\, f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x} \\ &\textit{subject to} \qquad A\underline{x} = \underline{b} \qquad\qquad (1) \\ &\textit{and} \qquad\qquad \underline{x} \geq 0 \end{aligned}$$

where c is a row vector with n components, D is an $n \times n$ symmetric matrix, b an m-vector and A is an $m \times n$ coefficient matrix.

If D is positive semi definite or more precisely, if f is a convex function over the convex set of feasible solutions

$$S = \{\underline{x} \mid A\underline{x} = \underline{b}, \underline{x} \geq 0\}, \tag{2}$$

Then (1) is called a convex quadratic programming problem.

The hyper surface given by $A\underline{x} = \underline{b}$ and $\underline{x} = 0$ are called boundary surfaces of the convex constraint set. The convex quadratic program is feasible if S is not empty. A feasible point x is a boundary point if it lies on at least one of the boundary hyper surfaces. Else, if $g_i(\underline{x}) < b_i$ and $x_i > 0$ for all i,j, then it is an interior point of the convex constraint set.

The convex quadratic program is solvable if $f(\underline{x})$ is bounded over S and achieves its minimum in S. A feasible point x^* that minimizes $f(\underline{x})$ is a solution or optimal point. i.e. $f(\underline{x}^*) \leq f(\underline{x})$ for all x belongs to S.

If S is closed, bounded and nonempty, then there exists at least one solution. If S is not bounded, the boundedness of $f(\underline{x})$ over S is not enough for a convex quadratic program to be solvable.

Concave Quadratic Programming

The concave quadratic programming is an important class of mathematical programming problems in which the objective function is quadratic and concave and the constraints are linear. The objective function may be the sum of a linear and a concave quadratic form and hence is concave.

Obtaining a global minimum to a concave program becomes easier than minimization of a general quadratic function due to the following theorem:

Theorem: Let $\bar{x}$ be the global minimum to the concave quadratic programming problem:

$$\min f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x} \tag{1}$$

$$subject\ to\ x \in S, \text{ where } S = \{\underline{x} | A\underline{x} = \underline{b}, \underline{x} \geq 0\},$$

where the matrix A is $m \times n$ $(m \leq n)$, $D = D'$ and c & b are n & m-column vectors respectively. We assume that S is a nonempty compact polyhedral set. Then $\bar{x}$ is an extreme point of S, Charnes and Cooper (1961).

Proof: Let x_j for j =1,..., m be the extreme points of S.

Since S is compact, f takes on its minimum at $\bar{x} \in S$.

If $\bar{x} \in J = \{j | j = 1, \dots, m, x_j \in S\}$, then $\bar{x}$ is an extreme point of S and the theorem is proved. Otherwise, $\bar{x}$ can be expressible as $\bar{x} = \sum_{j=1}^{m} \lambda_j x_j$, where $\lambda_j > 0, \sum_{j=1}^{m} \lambda_j = 1$ and $x_j \in J$. Since f is concave, it follows that

$$f(x) = f\left(\sum_{j=1}^{m} \lambda_j x_j\right) \geq \sum_{j=1}^{m} \lambda_j f(x_j). \tag{2}$$

But since $\bar{x}$ is the global minimum to (1) it follows that $f(\bar{x}) \leq f(x_j)$ for $\in J$. Then the relation (2) implies that $f(\bar{x}) = f(x_j)$ for $j \in J$. Thus $\bar{x}$ is an extreme point of S.

Duality in Quadratic Programming Problem

The role of dual programs in QPP is not as significant as in linear programming due to the lack of symmetry. The dual quadratic programs can be obtained by using K-T conditions.

Consider the QPP:

$$\text{maximize } f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}$$
$$\textit{subject to } A\underline{x} = \underline{b} \tag{1}$$
$$\textit{and } \underline{x} \geq 0.$$

We call (1) as the primal problem.

The Lagrangian form associated with (1) is

$$\phi(\underline{x}, \underline{\lambda}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x} + \underline{\lambda}'(\underline{b} - A\underline{x}).$$

Applying K-T conditions to (1) we get:

(i) $\nabla_{\underline{x}}\, \phi(\underline{x}, \underline{\lambda}) \leq 0 ==> A'\underline{\lambda} - 2D\underline{x} \geq 0,$

(ii) $\underline{x}'\nabla_{\underline{x}}\, \phi(\underline{x}, \underline{\lambda}) = 0 ==> \underline{c}'\underline{x} + \underline{x}'D\underline{x} = \underline{\lambda}'b - \underline{x}'D\underline{x},$

(iii) $\underline{x} \geq 0,$

(iv) $\nabla_{\underline{\lambda}}\, \phi(\underline{x}, \underline{\lambda}) = 0 ==> A\underline{x} = \underline{b}.$

From (ii) and (iv) $\phi(\underline{x}, \underline{\lambda})$ can be written as

$$\phi(\underline{x}, \underline{\lambda}) = \underline{\lambda}'\underline{b} - \underline{x}'D\underline{x}.$$

The dual of the QPP (1) can now be defined as:

$$\text{minimize } \phi(\underline{x}, \underline{\lambda}),$$
$$\textit{subject to } \nabla_{\underline{x}}\, \phi(\underline{x}, \underline{\lambda}) \leq 0$$
$$\textit{and } \quad \underline{x} \geq 0.$$

which is equivalent to:

$$\text{Minimize } \underline{\lambda}'\underline{b} - \underline{x}'D\underline{x} = F(\underline{x}, \underline{\lambda}),$$
$$\textit{subject to } A'\underline{\lambda} - 2D\underline{x} \geq 0 \tag{2}$$

$$\text{and} \quad \underline{x} \geq 0.$$

If D is a null matrix the dual program agrees with the Duality in Linear Programming. It can be seen that:

(i) If the feasible domain is empty for one problem (primal/dual) then either the feasible domain is empty for the other problem or the objective function is not bounded over the feasible domain. If the objective function is not bounded over the feasible domain for one problem, then the feasible domain is empty for the other problem.

(ii) If the primal problem has the solution then the dual also has a solution and the optimal values of the two objective functions are same. The converse is true only if D is strictly definite.

Chapter - Two

Methods for solving QPP

When the objective function $f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}$ in QPP is convex and to be minimized, the QPP is called a convex QPP, on the other hand when $f(\underline{x})$ is concave and it is to be minimized, the QPP is called a concave QPP. Kunzi, Krelle and Oettli (1966) discussed various methods for solving convex QPP. The most popular of these are Beale (1959) and Wolfe (1959) methods. In both the methods simplex algorithm is used and they are applicable to QPP where $\underline{x}'D\underline{x}$ is positive semidefinite. Rosen (1960) gave his method of gradient projection in which he used the projection of the gradient of the objective function on the boundary of the feasible domain and proceeded in its direction to improve the solution.

Tui (1964) gave a procedure for solving concave minimization problems with linear constraints which can be applied to concave QPP. Ritter (1965, 1966) developed a procedure to solve non-convex minimization problems with linear constraints. The above procedures are further improved by Cottle and Mylander (1970) and Zwart (1974). Arshad, Khan and Ahsan (1981) developed an algorithm for solving concave QPP using Tui (1964) cuts, in which at each iteration an approximate LPP is solved which gives an upper bound for the original problem.

Due to the limitation of space it is not possible to discuss all the available methods in this dissertation. Therefore, in the following sections discussions are limited to only few of them obviously the important ones.

Wolfe's Method for Solving a Quadratic Programming

Consider the QPP as:

$$maximize(or\ minimize) Q(x) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}$$

$$subject\ to \qquad A\underline{x} \leq \underline{b}$$

$$and \qquad \underline{x} \geq \underline{0}$$

Applying Kuhn-Tucker conditions to the above QPP we get the following set of linear and nonlinear equations and the nonnegativity restrictions:

$$-2D\underline{x} + A'\underline{\lambda} - \underline{u} = \underline{c} \qquad (1)$$

$$A\underline{x} + I\underline{s} = \underline{b} \qquad (2)$$

$$\underline{x}, \underline{\lambda}, \underline{u}\ \&\ \underline{s} \geq \underline{0} \qquad (3)$$

$$and\ \underline{u}'\underline{x} = \underline{0}\ ;\ \ \underline{\lambda}'\underline{s} = \underline{0} \qquad (4)$$

Our aim is to find a basic solution to the system of equations (1)-(2) with (3) that satisfies (4) also. The $\underline{x}$ part of this solution will solve the QPP. The required basic solution can be obtained by 'Phase-I' of the "Two-phase Simplex Method". The entries in the basis are restricted by the following rule to maintain the conditions imposed in (4).

"Rule: If u_j is in the basis at a positive level, x_j cannot becom basic at positive level. Similarly λ_i and s_i cannot be positive simultaneously." If 'Phase-I' of the 'Two-phase Simplex Method' fails to provide a basic solution to the system (1)-(3) the given QPP will have no solution.

Numerical Example: Use Wolfe's method to solve the QPP:

$$maximize\ Q(\underline{x}) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

$$s.t. \qquad x_1 + 2x_2 \le 4$$

$$x_1, x_2 \ge 0$$

Solution: Comparing the given QPP with standard form "$maximize\ Q(x) = \underline{c}'\underline{x} + \underline{x}'D\underline{x},\ s.t. A\underline{x} \le \underline{b},\ and\ \ \underline{x} \ge \underline{0}$" we get the parameters as:

$\underline{c}' = (4{,}6)$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$, $A = (1 \quad 2)\ and\ \ \underline{b} = (4)$.

The system of equations

$$-2D\underline{x} + A'\underline{\lambda} - \underline{u} = \underline{c}$$

$$A\underline{x} + I\underline{s} = \underline{b}$$

$$\underline{x}, \underline{\lambda}, \underline{u}\ \&\ \underline{s} \ge \underline{0}$$

$$and\ \underline{u}'\underline{x} = \underline{0}\ ;\ \ \underline{\lambda}'\underline{s} = \underline{0}$$

can now be expressed as:

$$-2\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\lambda_1 - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$(1 \quad 2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + s_1 = 4$$

$$x_1, x_2, \lambda_1, u_1, u_2\ \&\ s_1 \ge 0$$

$$u_1x_1 = u_2x_2 = 0 = \lambda_1 s_1$$

or $2x_1 + \lambda_1 - u_1 = 4$ (1)

$6x_2 + 2\lambda_1 - u_2 = 6$ (2)

$x_1 + 2x_2 + s_1 = 4$ (3)

$x_1, x_2, \lambda_1, u_1, u_2\ \&\ s_1 \ge 0$ (4)

$$u_1x_1 = u_2x_2 = 0 = \lambda_1 s_1 \tag{5}$$

We have to find a basic solution to the system (1)-(4) that satisfies (5) also. The $\underline{x}$ part of this basic solution will solve the QPP.

To apply the 'Phase-I' of the "Two-phase Simplex Method" we need two artificial variables $a_1, a_2 \geq 0$,(say). The artificial linear programming problem (LPP) will be as follows:

$$minimize \quad a_1 + a_2$$

$$s.t. \quad 2x_1 + \lambda_1 - u_1 + a_1 = 4$$

$$6x_2 + 2\lambda_1 - u_2 + a_2 = 6$$

$$x_1 + 2x_2 + s_1 = 4$$

$$x_1, x_2, \lambda_1, u_1, u_2, s_1, a_1, a_2 \geq 0$$

Obviously a starting basic feasible solution will be $a_1 = 4,\ a_2 = 6,\ s_1 = 4$ and all other variables equal to zero with basis matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The various simplex tableaus of Phase-I are given below:

Tableau '1'

Basic Var.	Present Value	x_1	x_2	λ_1	u_1	u_2	s_1	a_1	a_2	Ratio	
a_1	4	2	0	1	-1	0	0	1	0	-	
a_2	6	0	6	2	0	-1	0	0	0	6/6=1	$\longrightarrow$
s_1	4	1	2	0	0	0	1	0	1	4/2=2	
	10	2	6	3	-1	-1	0	0	0		

Present value of the obj. fun. $= \underline{c}' B^{-1} \underline{b}$

a_2 will leave the basis and x_2 will become a basic variable in its place. By usual transformation formula of Simplex method we get the next tableau as:

Tableau '2'

Basic Var.	Present Value	x_1	x_2	λ_1	u_1	u_2	s_1	a_1	a_2	Ratio
a_1	4	2	0	1	-1	0	0	1	0	4/2=2 →
x_2	1	0	1	1/3	0	-1/6	0	0	1/6	-
s_1	2	1	0	-2/3	0	1/3	1	0	-1/3	2/1=2
	4	2 ↑	0	1	-1	0	0	0	-1	

Now a_1 will leave the basis and x_1 will enter. The next tableau after usual transformation is:

Tableau '3'

Basic Var.	Present Value	x_1	x_2	λ_1	u_1	u_2	s_1	a_1	a_2
x_1	2	1	0	1/2	-1/2	0	0	1/2	0
x_2	1	0	1	1/3	0	-1/6	0	0	1/6
s_1	0	0	0	-7/6	1/2	1/3	1	-1/2	-1/3
	0	0	0	0	0	0	0	-1	-1

Since all $z_j - c_j \leq 0$ and all the artificial variables are nonbasic the solution provided by the 'Tableau 3' is the required basic solution to the system (1)-(4)

satisfying (5) also. Therefore the $\underline{x}$ part of this basic solution will solve the given QPP. The solution is $x_1^* = 2, x_2^* = 1$ *and* $Q^* = 4 \times 2 + 6 \times 1 - 2^2 - 3 \times 1^2 = 7$.

Beale's Method for Solving a Quadratic Programming

Consider the QPP as:

$$\left.\begin{array}{lcr} \text{Maximize } Z & = & \underline{c}'\underline{x} + \underline{x}'D\underline{x} \\ \textit{subject to} & & A\underline{x} \le \underline{b} \\ \textit{and} & & \underline{x} \ge \underline{0} \end{array}\right\} \quad (1)$$

It is assumed that the objective function Z is concave (convex for minimization) and a basic feasible solution to the above QPP exists and is known. It is also assumed that all the basic feasible solution to QPP (1) is nondegenerate.

Let $\underline{x}_B = \begin{pmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{pmatrix}$ denote the vector of basic variables and B denote the corresponding basis matrix (B will contain m columns of matrix A of $A\underline{x} = \underline{b}$ corresponding to current basic variables). Let the $m \times n$ coefficient matrix A and the n-component decision vector $\underline{x}$ be partitioned as:

$A = (B|R)$ and $\underline{x} = \begin{pmatrix} \underline{x}_B \\ \underline{x}_R \end{pmatrix}$ where R is an $m \times (n - m)$ matrix containing columns of A not in B and $\underline{x}_R$ is the vector of current non-basic variables.

The constraint equations $A\underline{x} = \underline{b}$ can now be expressed as:

$$(B|R)\begin{pmatrix} \underline{x}_B \\ \underline{x}_R \end{pmatrix} = \underline{b}$$

$$\text{or } B\,\underline{x}_B + R\underline{x}_R = \underline{b}$$

$$\text{or } \quad B\underline{x}_B = \underline{b} - R\underline{x}_R$$

$$\text{or } \quad B^{-1}\,B\underline{x}_B = B^{-1}\underline{b} - B^{-1}R\underline{x}_R$$

$$\text{or } \quad \underline{x}_B = B^{-1}\underline{b} - B^{-1}R\underline{x}_R$$

$$= \underline{y}_0 - \left(\underline{y}_1, \underline{y}_2, \dots, \underline{y}_j, \dots, \underline{y}_{n-m}\right)\underline{x}_R \qquad (2)$$

where $B^{-1}\underline{b} = \underline{y}_0 = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{pmatrix}$ and $\underline{y}_j = \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{pmatrix}; j = 1,2, \dots, (n-m)$

is the j^{th} column of the $m \times (n - m)$ matrix $B^{-1}R$.

(2) => $\qquad x_{Bi} = y_{i0} - \sum_{j=1}^{n-m} y_{ij}\, x_{Rj}\,; \;\; i = 1,2, \dots, m \qquad$ (3)

where x_{Bi} is the i^{th} component of $\underline{x}_B$ and x_{Rj} is the j^{th} component of $\underline{x}_R$.

Using (3) the basic variables can be eliminated from the objective function $Z = \underline{c}'\underline{x} + \underline{x}'D\underline{x}$ and it can be expressed as a function of nonbasic variables $x_{Rj}\,; j = 1,2, \dots, n - m$ alone as:

$$Z = Z_0 + \underline{\alpha}'\underline{x}_R + \underline{x}_R'G\underline{x}_R \qquad (4)$$

where $Z_0 = \textit{constant term}$

$\underline{\alpha}'\underline{x}_R = \textit{linear part}$

$\underline{x}_R'G\underline{x}_R = \textit{quadratic part}$

The present basic feasible solution $\underline{x}_B$ and the corresponding value of the objective function can be obtained directly by putting $x_{Rj} = 0; j = 1,2, \dots, n - m$ in (3) and (4) respectively as:

$$x_{Bi} = \underline{y}_{i0} \; ; i = 1,2, \dots, m \tag{5}$$

and $Z_{\underline{x}_B} = Z_0$ (6)

Now using (4) the rate of change of Z with respect to x_{Rj} can be obtained as:

$$\frac{\partial Z}{\partial x_{Rj}} = \alpha_j + 2\sum_{k=1}^{n-m} g_{jk}\, x_{Rk} \tag{7}$$

where α_j is the j^{th} element of $\underline{\alpha}$ and g_{jk} is the $(j, k)^{th}$ element of the matrix G.

At the present solution $x_{Rj} = 0$ hence $\frac{\partial Z}{\partial x_{Rj}} = \alpha_j$. Now it will pay to increase the value of x_{Rj} from zero to a positive level if

$$\frac{\partial Z}{\partial x_{Rj}} = \alpha_j > 0.$$

The value of x_{Rj} can be increased up to a level where (i) Any of the present basic variables vanishes or (ii) The rate $\frac{\partial Z}{\partial x_{Rj}}$ vanishes.

Let the basic variable x_{Bi} vanishes at $x_{Rj} = x_{Rj}^{(1)}$ and $\frac{\partial Z}{\partial x_{Rj}}$ vanishes at $x_{Rj} = x_{Rj}^{(2)}$.

An increase in the value of x_{Rj} beyond $x_{Rj}^{(1)}$ will make x_{Bi} negative and an increase in x_{Rj} beyond $x_{Rj}^{(2)}$ will result in a decrease in the value of Z, the objective function. Thus the desired value of x_{Rj} is given by

$$x_{Rj} = min\left\{x_{Rj}^{(1)}, x_{Rj}^{(2)}\right\} \tag{8}$$

We may have the following two possible cases:

<u>Case 1</u>: $minimum\,\{x_{Rj}^{(1)}, x_{Rj}^{(2)}\} = x_{Rj}^{(1)}$

⇨ A basic variable vanishes first (say x_{Bs})

⇨ x_{Rj} can be made basic with a value $x_{\mathrm{Rj}}^{(1)}$ and x_{Bs} will become nonbasic.

Thus a new improved basic feasible solution is obtained whose value of the objective function is greater than the previous one. The vectors $\underline{y}_0, \underline{\alpha}, \underline{y}_j$ the matrix G and the constant Z_0 are recomputed for the new improved solution.

<u>Case 2</u>: $minimum\ \{x_{Rj}^{(1)}, x_{Rj}^{(2)}\} = x_{Rj}^{(2)}$

⇨ The rate $\frac{\partial Z}{\partial x_{Rj}}$ vanishes first.

In this case we will have a new improved solution with $(m + 1)$ variables at positive level. To make this solution basic we add a new (dummy) constraint

$$u_j = \frac{\partial Z}{\partial x_{R_j}} \tag{9}$$

As $\frac{\partial Z}{\partial x_{\mathrm{R_j}}} = 0$ for the present solution $\mathrm{u_j}(= 0)$ will act as an additional nonbasic variable.

Equations (7) and (9) give

$$u_j = \alpha_j + 2\sum_{k=1}^{n-m} g_{jk}\, x_{R_k}$$

$$\text{or } u_j = \alpha_j + 2g_{jj}x_{R_j} + \sum_{\substack{k=1\\k\neq j}}^{n-m} g_{jk}\, x_{R_k}$$

$$\text{or } 2g_{jj}x_{R_j} = u_j - \alpha_j - 2\sum_{\substack{k=1\\k\neq j}}^{n-m} g_{jk}\, x_{R_k}$$

$$\text{or } x_{R_j} = \frac{1}{2g_{jj}}\left[u_j - \alpha_j - 2\sum_{\substack{k=1\\k\neq j}}^{n-m} g_{jk}\, x_{R_k}\right] \tag{10}$$

Using (10) x_{R_j} may be eliminated from (3) and (4) to express the basic variables and the objective function Z in terms of present nonbasic variables alone, that is the new values of vectors $\underline{y}_0, \underline{\alpha}$ and $\underline{y}_j$ and the matrix G and the constant Z_0 are recomputed. The new rates $\frac{\partial Z}{\partial x_{Rj}}$ and $\frac{\partial Z}{\partial u_j}$ (if any) are worked out and the method proceeds further to improve the present solution.

Termination Criterion: The procedure will terminate when $\frac{\partial Z}{\partial x_{Rj}} \leq 0$ for all nonbasic variables (For a minimization QPP $\frac{\partial Z}{\partial x_{Rj}} \geq 0$) and $\frac{\partial Z}{\partial u_j} = 0$ for all u_j (free variable).

Notes:

- As the method starts with a basic feasible solution to the constraint equations in $A\underline{x} = \underline{b}$ with $\underline{x} \geq 0$, the possibility of no solution is ruled out.
- The additional nonbasic variables u_j are termed as free variables because they may assume any value zero, negative or positive.
- Whenever possible the free variables are varied first for determining the next improved solution.
- While considering $\frac{\partial Z}{\partial u_j}$, both increase and decrease in u_j are to be examined because u_j is a free variable.
- At any stage if u_j becomes basic it need not to be considered any further.
- As the computations proceed the size of the basis may increase and may decrease to m again.
- While making a nonbasic variable basic if neither any basic variable nor the rate vanishes then the given quadratic programming problem will have an unbounded solution.
- When the termination criterion is satisfied the corresponding solution will give a local maximum of Z over the feasible region. As

$Z(x)$ is assumed to be concave this local maximum will also be global.

Numerical Example: Use Beale's Method to solve the following Quadratic Programming Problem:

$$\text{Maximize } Q(\underline{x}) = 10x_1 + 25x_2 - 10x_1^2 - x_1^2 - 4x_1x_2 \quad (1)$$

$$\textit{subject to} \quad x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 9$$

$$\textit{and} \quad x_1, x_2 \geq 0$$

Solution: Using x_3 and x_4 (≥ 0) as slack variables the given constraints can be expressed as:

$$x_1 + 2x_2 + x_3 = 10 \quad (2)$$

$$x_1 + x_2 + x_4 = 9 \quad (3)$$

Iteration 1: We have a starting solution as:

$\underline{x}^{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 9 \end{pmatrix}$, where x_1 & x_2 are n.b.v. and x_3 & x_4 are b.v.

Expressions (2) & (3) give the expressions of the b.v. in terms of n.b.v as:

$$x_3 = 10 - x_1 - 2x_2 \quad (4)$$

$$x_4 = 9 - x_1 - x_2 \quad (5)$$

The objective function (1) is already in terms of n.b.v x_1 &x_2.

The corresponding value of the objective function is $Q(\underline{x}^{(1)}) = 0$.

Optimality Check:

$$\frac{\partial Q}{\partial x_1} = 10 - 20x_1 - 4x_2 \;\; => \left(\frac{\partial Q}{\partial x_1}\right)_{\underline{x}^{(1)}} = 10 > 0$$

$$\frac{\partial Q}{\partial x_2} = 25 - 4x_1 - 2x_2 \quad => \left(\frac{\partial Q}{\partial x_2}\right)_{\underline{x}^{(1)}} = 25 > 0$$

Hence the optimality criterion is not satisfied.

Now $\{Max\left\{\frac{\partial Q}{\partial x_1}, \frac{\partial Q}{\partial x_2}\right\}_{\underline{x}^{(1)}} = Max\,\{10, 25\} = 25 = \left(\frac{\partial Q}{\partial x_2}\right)_{\underline{x}^{(1)}}$

⇨ x_2 is to be made basic.

While making x_2 basic:

$$\left.\begin{array}{r} x_3 \; vanishes \; at \; x_2 = 5 \\ x_4 \; vanishes \; at \; x_2 = 9 \\ \dfrac{\partial Q}{\partial x_2} \; vanishes \; at \; x_2 = 12.5 \end{array}\right\}$$

x_3 vanishes first. We have Case 1. x_3 is to be made non basic.

<u>Iteration 2</u>: Expressing b.v. x_2 & x_4 interms of n.b.v. x_1 & x_3 we get:

By (4) $\qquad 2x_2 = 10 - x_1 - x_3$

or $\qquad x_2 = 5 - \frac{1}{2}x_1 - \frac{1}{2}x_3 \qquad (6)$

By (5) and (6)

$$x_4 = 9 - x_1 - \left(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right)$$

$$= 9 - x_1 - 5 + \frac{1}{2}x_1 + \frac{1}{2}x_3$$

or $x_4 = 4 - \frac{1}{2}x_1 + \frac{1}{2}x_3 \qquad (7)$

Expressing the objective function intents of n.b.v. we get:

$$Q(x_1, x_3) = 10x_1 + 25\left(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right) - 10x_1^2 - 5(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3)^2$$
$$- 4x_1(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3)$$

$$= 10x_1 + 125 - \frac{25}{2}x_1 - \frac{25}{2}x_3 - 10x_1^2 - 25 + 5x_1 + 5x_3 -$$
$$\frac{1}{2}x_1x_3 - \frac{1}{4}x_1^2 - \frac{1}{4}x_3^2 - 20x_1 + 2x_1^2 + 2x_1x_3$$

or $Q(x_1, x_3) = 100 - \frac{55}{2}x_1 - \frac{15}{2}x_3 - \frac{33}{4}x_1^2 - \frac{1}{4}x_3^2 + \frac{3}{2}x_1x_3$ (8)

Thus the new improved solution is given by:

$\underline{x}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \\ 4 \end{pmatrix}$ with a value of the objective function $Q(\underline{x}^{(2)}) = 100$

<u>**Optimality Check**</u>:

$$\frac{\partial Q}{\partial x_1} = -\frac{55}{2} - \frac{33}{2}x_1 + \frac{3}{2}x_3 \quad => \left(\frac{\partial Q}{\partial x_1}\right)_{\underline{x}^{(2)}} = -\frac{55}{2} < 0$$

$$\frac{\partial Q}{\partial x_3} = -\frac{15}{2} - \frac{1}{2}x_3 + \frac{3}{2}x_1 \quad => \left(\left(\frac{\partial Q}{\partial x_3}\right)_{\underline{x}^{(2)}} = -\frac{15}{2} < 0\right)$$

⇨ Optimality criterion is satisfied.

Thus the required optimal solution to the given quadratic programming solution is

$x_1^* = 0,\ x_2^* = 5\ \ and\ \ Q^* = 100.$

Solution of a Concave Quadratic Programming Problem

Arshad, Khan and Ahsan (1981) developed the following algorithm for solving a concave QPP. Consider the problem

$$\left.\begin{array}{ll}\text{Minimize} f(\underline{x}) = & \underline{c}'\underline{x} + \underline{x}'D\underline{x} \\ \textit{subject to} & A\underline{x} = \underline{b} \\ \textit{and} & \underline{x} \geq \underline{0}\end{array}\right\} \tag{1}$$

where $\underline{x}'D\underline{x}$ is negative semi definite.

Let S denote the set of all feasible solutions to QPP (1), that is

$$S = [\underline{x} \mid A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}] \tag{2}$$

It is assumed that S is bounded.

The number of local minima of (1) may be large. The interest lies in finding the global minimum of (1).

The extreme points of S can be written as:

$$x_1 = a_{10} + \sum_{j=1}^{n'} a_{1j}\left(-y_j\right) \geq 0,$$

$$\vdots$$

$$x_m = a_{m0} + \sum_{j=1}^{n'} a_{mj}\left(-y_j\right) \geq 0,$$

$$x_{m+1} = 0 + y_1 \qquad \geq 0,$$

$$\vdots$$

$$x_n = 0 + y_n \qquad \geq 0,$$

where $n' = n - m$ and $y_1, y_2, \dots, y_{n-m}$ are the present non-basic variables. The above equations can be combined as:

$$\underline{x} = \underline{a}_0 + \sum_{j=1}^{n'} \underline{a}_j\left(-y_j\right) \geq 0 \tag{3}$$

where $\underline{x}' = (x_1, x_2, \dots, x_n), \underline{a}_0' = (a_{10}, \dots, a_{m0}, 0, \dots, 0)$ and $\underline{a}_j' = (a_{1j}, \dots, a_{mj}, 0, \dots, 0), \underline{a}_0$ and $\underline{a}_j$ both have their last $(n - m)$ components equal to zero.

Let f_p denote the value of the objective function $f(\underline{x})$ in (1) at the present local minimum.

Consider n′ maximization problems

$$\begin{aligned} &Maximize\ y_j \\ &subject\ to\ f(\underline{a}_0 + \underline{a}_j)(-y_j) \geq f_p, \\ &\qquad 0 \leq y_j \leq \bar{y}, j = 1, \dots, n', \end{aligned} \tag{4}$$

where $\bar{y}$ is some large number.

Let y_j^* be the non-degenerate optimal solution of the j^{th} problem in (4), that is, $y_j^* > 0, j = 1, \dots, n'$. The cut

$$T_1: \sum_{j=1}^{n'} \frac{y_j}{y_j^*} \geq 1, \tag{5}$$

First used by Tui (1964). Eliminates only that portion of the feasible set S which does not contain any solution better than the present local minimum of $f(\underline{x})$.

Charnes and Cooper (1961) showed that for a concave objective function the global minimum lies on an extreme point of the feasible region. Let $\underline{x}^{\mathrm{m}}$ be an extreme point of the set

$$S_1 = S = [\underline{x} | A\underline{x} = b, \underline{x} \geq 0]. \tag{6}$$

Consider the linear function

$$L_m(\underline{x}) = \nabla' f(\underline{x}^m)\underline{x} \tag{7}$$

where $\nabla f(\underline{x}^m)$ is the value of the gradient vector of $f(\underline{x})$ at $\underline{x} = \underline{x}^m$.

From concavity of $f(\underline{x})$ we have

$$L_m(\underline{x}) \geq f(\underline{x}) \quad \text{for all } x. \tag{8}$$

Let $\underline{x}^m$ denote a local minimum of QPP (1) and $\underline{x}^{L_1}$ denote the solution to the LPP:

$$\begin{aligned} &Minimize\ L_{m_1}(\underline{x}) = \nabla' f(\underline{x}^{m_1})\underline{x}, \\ &subject\ to\ \ \underline{x} \in S_1. \end{aligned} \tag{9}$$

Clearly $\underline{x}^{L_1}$ will provide an upper bound to the optimal solution to the problem (1). Taking $\underline{x}^{L_1}$ as starting point, the problem is to locate another local minimum of (1).

This could be done by moving along the various binding edges of S_1. The search will end with an extreme point $\underline{x}^{m_2}$ of S_1 such that $f(\underline{x}^{m_2})$ is the minimum of $f(\underline{x})$ over all extreme points adjacent to $\underline{x}^{m_2}$. The whole procedure is then repeated with the following LPP:

$$\begin{aligned} &Minimize\ L_{m_2}(\underline{x}) = \nabla' f(\underline{x}^{m_2})\underline{x}, \\ &subject\ to\ \ \underline{x} \in S_2. \end{aligned} \tag{10}$$

where $S_2 = S_1$ except when a Tui's cut (5) is introduced into the constraints, in which case $S_2 = S_1 \cap T_1$.

The process will terminate at k^{th} iteration if the $(k+1)^{th}$ LPP:

$$\begin{aligned} &Minimize\ L_{m_{(k+1)}}(\underline{x}) = \nabla' f(\underline{x}^{m_{(k+1)}})\underline{x}, \\ &subject\ to\ \ \underline{x} \in S_{(k+1)}. \end{aligned} \tag{11}$$

has no solution. The extreme point $\underline{x}^{m_k}$ will then be the global minimum for QPP (1).

Remark: When Tui's cuts are introduced at any iteration k, S_{k+1} will be the intersection of S_k with T_k. The theorem may not remain valid. However, it is known that the cuts eliminate the current local minimum without eliminating the absolute minimum.

Numerical Illustration

Arshad, Khan and Ahsan solved the following example for illustrating the computational details.

$$Minimize\ f(\underline{x}) = 3x_1 + 4x_2 - x_1^2 - 2x_2^2, \tag{12}$$

$$\begin{aligned} subject\ to \qquad & -2x_1 + x_2 \leq 1, \\ & -x_1 + x_2 \leq 2, \\ & x_2 \leq 4, \\ & x_1 + x_2 \leq 7, \\ & x_1 - x_2 \leq 3, \\ & x_1 - 2x_2 \leq 2 \\ \text{and } & x_1, x_2 \geq 0. \end{aligned} \tag{13}$$

Origin can be taken as the starting basic feasible solution, that is,

$$\underline{x}^0 = (0,0).$$

The value of the objective function at $\underline{x}^0$ is

$$f(\underline{x}^0) = 0.$$

Iteration 1: (0, 1) and (2, 0) are the two extreme points adjacent to $\underline{x}^0$. The value of the objective function (12) at both these points is 2 which is greater than $f(\underline{x}^0)$. Therefore the first local minimum $\underline{x}^{m_1} = \underline{x}^0$. For obtaining Tui's cut the following two one variable optimization problems are solved.

$$Maximize \quad x_1,$$
$$subject\ to \ \ 3x_1 - x_1^2 = 0, x_1 \geq 0, \tag{14}$$

and $Maximize \quad x_2,$
$$subject\ to \ \ 4x_2 - 2x_2^2 = 0, x_2 \geq 0, \tag{15}$$

The optimal solution of (14) and (15) are $x_1 = 3$ and $x_2 = 2$ respectively. Therefore the Tui's cut to be introduced is

$$T_1 : \geq \frac{x_1}{3} + \frac{x_2}{2} \geq 1, \tag{16}$$

The first LPP is:

$$Minimize\ L_{m_1}(\underline{x}) = f'(\underline{x}^{m_1})\underline{x} = 3x_1 + 4x_2,$$
$$subject\ to \ \ (13)\ and\ (16). \tag{17}$$

The solution to (17) is

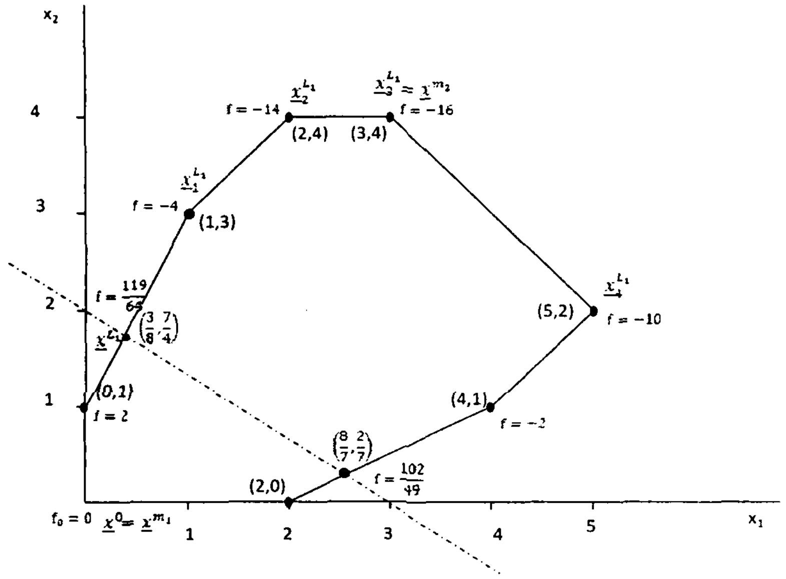
$$\underline{x}^{L_1} = \left(\frac{3}{8}, \frac{7}{4}\right), \tag{18}$$

with $f(\underline{x}^{L_1}) = \frac{119}{64}$.

<u>Iteration 2</u>: From $\underline{x}^{L_1}$ we move successively to the adjacent extreme points $x_1{}^{L_1}, x_2{}^{L_1}, x_3{}^{L_1}$ and $x_4{}^{L_1}$ with corresponding values of the objective function as -4, -14, -16 and -10 respectively (see figure)

$$\underline{x}^{m_2} = x_3{}^{L_1} = (3, 4). \tag{19}$$

The second LPP with the new Tui's cut as additional constraint is seen to be infeasible. Thus $\underline{x}^{m_2}$ is the optimum solution to the problem (12)-(13).



Some Other Methods for Solving Quadratic Programming Problem

Beside the three methods discussed in detail in this dissertation there are so many other methods for solving a QPP. In the following a brief account of some other well known methods for QPP has been given.

Hildreth (1957) has developed a method for solving a convex QPP which makes use of duality. The computations in this method are very simple but the convergence is very poor and usually a great number of iterations are needed. The objective function is also required to be strictly convex.

The method due to Theil and Van de Panne (1961) is also limited to strictly convex objective functions. In this method systems of equations satisfied by the constraints are used to determine all basic feasible solutions. The basic feasible solution with the optimal value gives the solution to the problem.

The method of Brankin and Dorfman (1956) is similar to that of Wolfe which partly uses a new algorithm which is not always successful.

Frank and Wolfe (1956) gave an improved of the Barankin-Dorfman method which uses the simplex technique.

Rosen (1960, 1961) gave Rosen's Gradient Projection Method for solving a general non-linear programming problem, but the method is seen to be more effective for a nonlinear programming problem with linear constraints, specially for quadratic programming problems.

Frisch (1957) has developed a method known as Multiplex method. This method is similar to that of Rosen (1960, 1961).

Zoutendjik (1960) gave the method of feasible directions. Houthakker (1960) gave a method known as capacity method for solving QPP. The method is applicable only under restricted hypotheses regarding the constraint and for a strictly convex objective function which limits it's utility.

Beale (1967) extended the 'Inverse Matrix Method' of linear programming to solve convex QPP.

Chapter - Three

Integer Quadratic Programming Problems

Introduction

A QPP in which all or some variables are restricted to be integers is called an integer QPP. The mathematical model of an integer QPP can be given as:

$$
\begin{aligned}
&\text{maximize(or minimize) } f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}, \\
&\textit{subject to} \qquad A\underline{x} = \underline{b}, \qquad\qquad (1) \\
&\qquad\qquad\qquad\quad \underline{x} \geq \underline{0}, \\
&\textit{and} \quad x_j \textit{ is an integer if } j \in J
\end{aligned}
$$

where $J = [j|x_j$ is required to be integer], that is, J is the set of indices of those variables which are required to be integers.

Clearly if $\underline{x} = (x_1, x_2, \ldots, x_n)$ then we will have an all integer quadratic programming problem if

$$J = [1, 2, \ldots, n] = I(say). \qquad (2)$$

On the other hand if $J \subset I$, that is, is some proper subset of I we will have a mixed integer quadratic programming problem.

Kunzi and Oettli (1963) first considered the problem of integer QPP and gave their method for solving all integer QPP using cutting plane technique. Later Agrawal (1974a) gave an algorithm for solving the same problem by making use of branch and bound technique. Bari and Arshad (1978) gave a variation of Agrawal's algorithm. Agrawal (1974b) also used Gomory's cuts for solving mixed integer QPP.

All Integer Quadratic Programming Problem

We will discuss, in this section the method of Agrawal for solving an all integer convex QPP with its variation presented by Bari and Arshad (1978).

Agrawal's method for solving all integer convex QPP: Agrawal used Beale's method for solving the convex QPP, without integer restrictions and then he used the branch and bound technique for obtaining the integer solution.

Consider the problem:

$$\begin{aligned} &\text{minimize} f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}, && (a) \\ &subject\ to \quad A\underline{x} = \underline{b}, && (b) \qquad\qquad (3) \\ &\underline{x} \geq \underline{0}, && (c) \end{aligned}$$

$$and \quad x_j\ is\ an\ integer\ for\ all\ j \in J = [1,2,\dots,n]\ (d)$$

Assume that $\underline{x}'D\underline{x}$ is positive semi definite so that $f(\underline{x})$ is a convex function of n variables $\underline{x} = (x_1, \dots, x_n)$. It is also assumed that the constraints (3b) are feasible, the feasible set is bounded and degeneracy is absent.

The Algorithm: Using Beale's method the problem (3a) to (3c) is first solved without restriction (3d). Let $x^0 \geq 0$ denote this solution and f_0 denote the value of $f(x^0)$. If all the components of x^0, that is, $x_1^0, x_2^0, \dots, x_n^0$ are integers $x^0 = x^*$ will be the required optimal solution. If some or all $x_j^0, j = 1, \dots, n$ are non-integers, Land and Doig (1960) method is used to obtain more restrictive lower bounds $f_1, f_2, \dots, f_k$ on $f(\underline{x})$.

Let x_p^0 denote any noninteger component of x^0. Then

$$[x_p^0] < x_p^0 < [x_p^0] + 1$$

where denote $[x_p^0]$ is the integral part in x_p^0. The two successive integers nearest to x_p^0 are $[x_p^0]$ and $[x_p^0]$+1. In order to have a solution of (3a)to (3c) with integer x_p the following two subproblems are solved again by Beale's method:

(1) Solve (3a) to (3c) with $x_p = [x_p^0]$ as an additional constraint.

(2) Solve (3a) to (3b) with $x_p = [x_p^0] + 1$ as an additional constraint.

Let the two values of the objective function obtained in (1) and (2) are respectively f' andf''.

If both the sub problems (1) and (2) have no solution this implies that x_p cannot have an integer value and the original problem (3) has no solution.

If the sub problem (1) has no solution this implies that for further consideration we must keep $x_p \geq [x_p^0] + 1$. Similarly if subproblem (2) has no solution for further consideration we have $x_p \leq [x_p^0]$.

Let $f_1 = \min(f', f'')$ and $x_p =$ h where h is an integer. To find the second best solution solve the two sub problems (3a) to (3c) with $x_p = h - 1$ and $x_p = h + 1$ separately. (One of these values has already been obtained as f' or f''). Let x_q be any non-integral variable at this stage. Solve again the two subproblems with $x_p = h$ and $x_q = [x_q^0]$ and $x_p = h$ and $x_q = [x_q^0] + 1$ as additional constraints. Let f_2 be the most minimum value of $f(\underline{x})$ obtained till now.

The procedure is then repeated with f_2 as the current lower bound on $f(\underline{x})$.

Continuing in the above manner a tree can be formed whose every vertex represent a known set of integer constraints. A branch of this tree will terminate if it reaches a vertex having non-feasible solution. At last either all branches will terminate or a vertex having the most minimum value is reached for which all x_j are integers.

The convergence of the algorithm can be seen in Agrawal (1974a); he also solved the following numerical example to illustrate the computational details.

A numerical example: Consider the convex QPP:

$$\begin{aligned} &\text{Min } f(\underline{x}) = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2, \\ &\textit{subject to } x_1 + x_2 \leq 2, \\ &\qquad x_1, x_2 \geq 0, \\ &\qquad x_1 \text{ and } x_2 \text{ integers.} \end{aligned} \tag{4}$$

The constraint $x_1 + x_2 \leq 2$ can be written as $x_1 + x_2 + x_3 = 2$ where x_3 is a slack variable. Application of Beale's method to solve (4) without integer restrictions given us:

$$x_1 = \frac{3}{2},\ x_2 = \frac{1}{2} \textit{ and } f = \frac{1}{2}.$$

Let us first consider x_1 in the above solution.

The two subproblem with $x_1 = 1$ and $x_1 = 2$ as additional restrictions yields

$$x_1 = 1, x_2 = \frac{1}{2}, x_3 = \frac{1}{2} \textit{ and } f = \frac{3}{2} \tag{5}$$

and $x_1 = 2, x_2 = 0, x_3 = 0 \textit{ and } f = 2$ (6)

Clearly solution (5) will provide a better lower bound on $f(\underline{x})$.

Further taking $x_1 = 0$ as an additional constraint gives:

$x_1 = 0, x_2 = 0, x_3 = 2$ and $f = 6$ which will be discarded because we have a better solution (5)

Thus $f_1 = \frac{3}{2}$.

The following sub problem is now solved:

$$\text{Min } f(\underline{x}) = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2,$$

$$\textit{subject to } x_1 + x_2 + x_3 = 2,$$

$$x_1 = 1,$$

$$x_2 = 0,$$

$$\text{and} \quad x_1, x_2, x_3 \geq 0.$$

The value of $f(\underline{x})$ comes out to be 2. Therefore $\underline{x} = (x_1^*, x_2^*) = (1,0)$ will be the required optimum integer solution to (4) with 2 as the minimum value of the objective function.

<u>Bari and Arshad's variation of Agrawal's method</u>:

Consider the QPP:

$$\begin{aligned} &\text{maximize } f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}, && (a) \\ &\textit{subject to } \; A\underline{x} = \underline{b}, && (b) \qquad (7) \\ &\underline{x} \geq \underline{0}, && (c) \\ &\text{and } x_j \text{ is an integer for all } j \epsilon I = (1,2,\dots,n) && (d) \end{aligned}$$

Assume that $\underline{x}'D\underline{x}$ is negative semi definite, that is, $f(\underline{x})$ is a concave function of x, A, D, b, c and x are as defined in (2). If the integer restrictions (3d) are neglected than an equivalent problem for (7a) to (7c) can be stated as:

$$\begin{aligned} &\text{Find vectors } (x,y,z) \geq 0, && \text{(a)} \\ &\text{such that } \; 2Dx - A'y + Iz = -c, && \text{(b)} \qquad (8) \\ &Ax = b && \text{(c)} \\ &\text{and} \quad x'z = 0 && \text{(d)} \end{aligned}$$

The first n components of any basic feasible solution to (8) will be the optimal solution for the QPP (7a) to (7c). If this optimal solution, x^0 is an integer solution the problem (7) is solved.

Let the j^{th} component of x^0 i.e. x_j^0 is not an integer. Denote by $[x_j^0]$ the integer part of x_j^0. Let $[x_q^0] = h$.

Consider the following restrictions on x_j

$$x_j \leq h$$

$$\text{or} \quad h - x_j \geq 0 \tag{9}$$

$$\text{and} \quad x_j \geq h + 1 = k\ (say)$$

$$\text{or} \quad x_j - k \geq 0. \tag{10}$$

The two new sub problems are created as follows:

(1) Solve (7a) to (7c) with (9) as an additional constraint. The K-T conditions for this problem are

$$(\bar{x}, y, z) \geq 0,$$

$$2Dx - A'y + Iz = -c, \tag{11}$$

$$Ax = b$$

$$\text{and} \quad x'z = 0$$

where $\bar{x} = x$ except that $\bar{x}_j = h - x_j$, and $\bar{I} = I$ except that the j^{th} diagonal element is -1, because $\frac{\partial f}{\partial x_j} = -\frac{\partial f}{\partial x_j}$.

(2) Solve (7a) to (7c) with (10) as an additional constraint.

The K-T conditions for this problem are:

$$(\bar{x}, y, z) \geq 0,$$

$$2Dx - A'y + Iz = -c, \tag{12}$$

$$Ax = b$$

and $$x'z = 0$$

where $\bar{x} = x$ except that now $\bar{x}_j = x_j - k$.

The sub problems (11) and (12) can be solved by simplex method. Dakin's approach is used in obtaining upper bounds on the solutions.

The convergence of the procedure is obvious because it uses Wolf's method.

The following numerical example will illustrate the computational details.

A numerical example: Consider the following all integer convex QPP:

$$\begin{aligned} &maximize\ f(\underline{x}) = 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2, && (a) \\ &subject\ to\ \ x_1 + x_2 \le 2, && (b) \\ &\qquad x_1, x_2 \ge 0, && (c) \\ &\qquad x_1 \text{ and } x_2 \text{ integers.} && (d) \end{aligned} \qquad (13)$$

Problem (13a) to (13c) is equivalent to:

Find $(x,y,z) \ge 0$,

such that $-4x_1 + 2x_2 - y + z_1 = -6,$

$$2x_1 - 4x_2 - y + z_2 = 0 \qquad (14)$$

and $x_1z_1 + x_2z_2 = 0.$

The solution to (14) obtained by artificial basis technique is

$$x_1 = \frac{3}{2}, x_2 = \frac{1}{2}\ and\ f = 1.$$

The value of the objective function is $\frac{5}{2}$.

The two subproblems can be obtained by adding $x_1 \le 1$ and $x_1 \ge 2$ in the constraints of (13). The equivalent set of K-T conditions as given in (11) and (12) are:

$$\bar{x}_1, x_2, y, z_1, z_2 \geq 0,$$

$$-4\bar{x}_1 - 2x_2 + y + z_1 = 2,$$

$$-2x_1 - 2x_2 + y + z_2 = -2, \tag{15}$$

$$-\bar{x}_1 + x_2 = 1$$

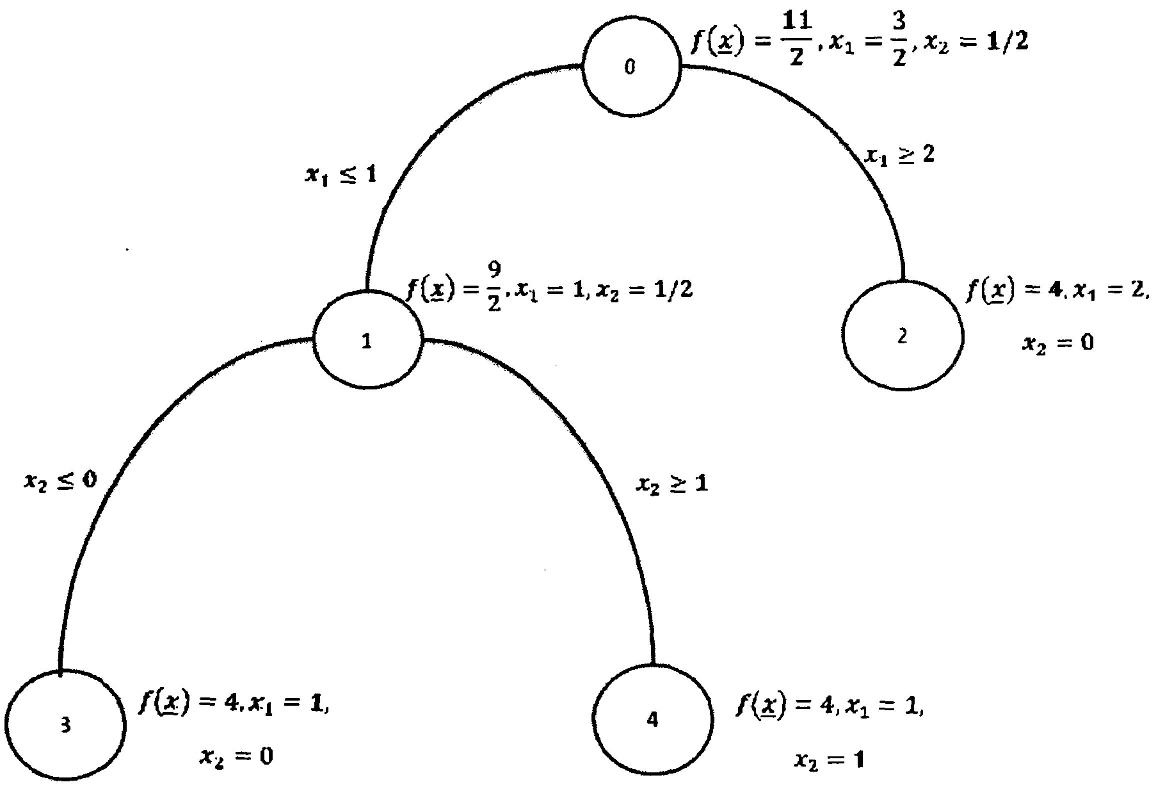
and $x_1 z_1 = 0, x_2 z_2 = 0.$

A solution to (15) is

$$\bar{x}_1 = 0, x_2 = \frac{1}{2} \; and \; z_1 = 3.$$

The other solutions of the sub problems can be obtained as above. The optimal solution of the above numerical example is not unique, however the maximum value of the function is 4.

The diagram below shows the various steps:



Mixed Integer Quadratic Programming Problem

In this section, we will describe a method for solving a convex QPP when only some, not all, variables are restricted to be integers. The method is due to Agrawal (1974b).

Agrawal's method for solving mixed integer convex QPP:

Agrawal used the cutting plane method introduced by Gomory (1958) for solving mixed integer convex QPP.

Consider the problem:

$$\begin{aligned} &\text{minimize} f(\underline{x}) = \underline{c}'\underline{x} + \underline{x}'D\underline{x}, && (a) \\ &subject\ to\ \ A\underline{x} = \underline{b}, && (b) \\ &\underline{x} \geq \underline{0}, && (c) \\ &\text{and } x_j \text{ is an integer for all } j\epsilon I && (d) \end{aligned} \qquad (16)$$

where $J = [j|x_j$ is an integer]. Assume that $\underline{x}'D\underline{x}$ is positive semi-definite, that is, $f(\underline{x})$ is convex. x, c, b, A and D are as defined in QPP (2). It is also assumed that the constraints (16b) and (16c) are feasible, the feasible set is bounded and degeneracy is absent.

The Algorithm: The problem (16a) to (16c) is first solved by Beale's method. Let us call $x_1, \dots, x_n$ as proper variables. During the application of Beale's method all the free variables which have been made basic at any stage should not be considered further. During variations of the nonbasic variables the equations of the proper variables are used to keep these basic free variables non-negative. Because in order to introduce Gomory type cuts all the variables, proper and free, should be non-negative. We will now call a free variable as improper variable as it is no more free. Let x denote the optimum solution to (16a)-(16c). Then we must have

$$\frac{\partial f}{\partial x_j} \geq 0, \text{ for all non-basic } x_j$$

and $\frac{\partial f}{\partial u_k} = 0$, for all free non-basic u_k

where u_k, $k = 1, \dots, h$ are the free variables at the final test point and the objective function $f(\underline{x})$ is in terms of only non-basic variables.

If all $x_j, j \in J$ are integers, x will be the optimum solution to (16).

Let all x_j for $j \in J$ are not integers, select any one of these x_j's say x_p. Then x_p can be expressed as

$$x_p = a_{p0} + \sum a_{pj}(-x_j) + \sum a_{pk}(-u_k), \tag{17}$$

where the two summations are for all nonbasic proper variables and non-basic improper variables. Clearly a_{p0} is non-integral.

Let us denote the integral and fractional parts of a_{p0}, a_{pj} and a_{pk} by $[a_{p0}], f_{p0}, [a_{pj}], f_{pj}$ and $[a_{pk}], f_{pk}$ respectively. Clearly we must have $0 < f_{p0} < 1, 0 < f_{pj} < 1$ *and* $0 < f_{pk} < 1$.

The Gomory cut can now be introduced as a basic variables S, where

$$S = -f_{p0} + \sum(-f_{pj})(-x_j) + \sum(-f_{pk})(-u_k) + \sum_{[j \notin J, a_{pj} \geq 0]} (-a_{pj})(-x_j)$$

$$+ \sum_{[j \notin J, a_{pj} < 0]} (f_{p0} a_{pj}) / (1 - f_{p0})(-x_j) + \sum_{[k \notin J, a_{pk} \geq 0]} (-a_{pk})(-u_k)$$

$$+ \sum_{[k \notin J, a_{pk} < 0]} (f_{p0} a_{pk}) / (1 - f_{p0})(-u_k). \tag{18}$$

The problem (16a) to (16c) can now be solved with (18) as an additional constraint by "parameter't' method" introduced by Beale (1959).

Define

$$s_t = s + t \tag{19}$$

where t is the Beale's parameter. Clearly the value of t for which the present solution is feasible is f_{p0}.

The parameter t method now gradually decreases the value of t to zero. If $t < f_{p0}$, then $s_t < 0$ and s_t will become nonbasic. If s_t contains any nonzero term in any improper variable this should be made basic. If such improper variable is not unique, any one of them could be chosen first. The process terminates when t=0 without any basic proper variables or any partial some non-integer values of the variables which are constrained derivative of f becoming negative. If at this stage we still have to be integers, more cuts are added one by one and the process is repeated until we reach the required optimal solution.

<u>**A numerical example**</u>: The following example will illustrate the computational details.

Consider the problem:

$$\begin{aligned}
&\text{Min}\, f(\underline{x}) = 183 - 44x_1 - 42x_2 + 8x_1^2 - 12x_1x_2 + 17x_2^2 && \text{(a)},\\
&\text{subject to} \quad 2x_1 + x_2 \le 10 && \text{(b)}, \quad (20)\\
&\qquad x_1, x_2 \ge 0 && \text{(c)},\\
&\text{and} \quad x_1 \text{ is an integer} && \text{(d)}.
\end{aligned}$$

Introducing x_3 as slack variable (20b) and (20c) can be written as

$$2x_1 + x_2 + x_3 = 10,$$

$$x_1, x_2, x_3 \ge 0.$$

The solution of the above problem by Beale's method yields:

$$x_1 = \frac{19}{5} + \frac{1}{5}u_2 - \frac{2}{3}x_3,$$

$$x_2 = \frac{12}{5} - \frac{2}{5}u_2 - \frac{1}{5}x_3,$$

$$f = 19 + 6x_3 + 3x_3^2 + 4u_2^2,$$

that is, $x_1 = \frac{19}{5}, x_2 = \frac{12}{5}$ and $f(x) = 19$.

Since x_1 is required to be integer, thus we have the Gomory's cut as

$$s = -\frac{4}{5} + \frac{\left(\frac{4}{5} \times -\frac{1}{5}\right)}{1 - \frac{4}{5}}(-u_2) + \frac{2}{5}x_3$$

or $$s = -\frac{4}{5} + \frac{4}{5}u_2 + \frac{2}{5}x_3. \qquad (21)$$

Addition of parameter t to (21) gives

$$s_1 = -\frac{4}{5} + t + \frac{4}{5}u_2 + \frac{2}{5}x_3.$$

s_1 will now become non-basic in place of u_2.

We have

$$u_2 = 1 - \frac{5}{4}t + \frac{5}{4}s_1 - \frac{1}{2}x_3,$$

$$x_1 = 4 - \frac{1}{4}t + \frac{1}{4}s_1 - \frac{1}{2}x_3,$$

$$x_2 = 2 + \frac{1}{2}t - \frac{1}{2}s_1$$

and $f = 19 + 6x_3 + 3x_3^2 + 4(1 - \frac{5}{4}t + \frac{5}{4}s_1 - \frac{1}{2}x_3)^2$.

t is now reduced to zero without making x_1 and x_2 or any partial derivatives of f negative which yields

$$x_1 = 4, x_2 = 2, \text{and } f = 23. \qquad (22)$$

(22) will be the required solution of the mixed integer QPP (20). The integral value of $x_2 = 2$, here is just by chance.

***Some other algorithms for mixed integer QPP*:**

Bala's (1969) developed a partitioning algorithm for solving all integer and mixed integer QPP which is based on his generalization of the dual systematic quadratic programs.

Recently Lazimy (1982) gave a method for solving general mixed integer QPP which uses the generalized Bender's decomposition algorithm developed by Geoffrion (1972). The original mixed integer QPP is decomposed into a series of all integer LPP and QPP without integer variables.

Chapter - Four

Applications of Quadratic Programming Problem

Introduction

Quadratic programming is a practical subject. In fact the rapid development of the subject is a result of many practical applications that have been found. In this chapter, we will outline some applications of QPP to certain problems arising in industry, economics and sample surveys.

The problem of optimal utilization of machine capacity

Let a manufacturing plant produces n commodities in quantities $x_1, x_2, \ldots, x_n$ units per period, using m machines available for $b_1, \ldots, b_m$ hours per period respectively.

Let a_{ij} denote the requirement of i^{th} machine hours for per unit production of j^{th} commodity. Thus we have the following m constraints:

$$\sum_{j=1}^{n} a_{ij}\, x_j \leq b_j, i = 1, \ldots, m \tag{1}$$

and also $x_j \geq 0, j = 1, \ldots, n$ (2)

Let the structure of the market is such that the per unit profit p_j on the j^{th} commodity depend on the quantities to be sold, that is,

$$P_j = p_j(x_1, \ldots, x_n)$$

The total profit P is then given as

$$P = \sum_{j=1}^{n} p_j = p_j(x_1, \ldots, x_n)x_j \tag{3}$$

Thus we can state the problem as:

$$\text{maximize } P = \sum_{j=1}^{n} p_j\,(x_1, \dots, x_n)x_j, \tag{4}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, i = 1, \dots, m$$

$$\text{and } x_j \geq 0 \quad j = 1, \dots, n$$

The problem (4) will become a QPP if p_j is a linear function of x_j. For example we can take the most general situation when p_j is proportional to x_j, that is,

$$p_j = k_j x_j$$

where k_j is some constant.

The total profit in which case is

$$P = \textstyle\sum_{j=1}^{n} k_j x_j^2 = k_1 x_1^2 + k_2 x_2^2 + \cdots + k_n x_n^2 \tag{5}$$

which is a quadratic function $\underline{x}' D \underline{x}$ of $\underline{x} = (x_1, \dots, x_n)$ where D is a diagonal matrix of order $n \star n$ whose j^{th} diagonal element is k_j.

The problem of maximization of (4) subject to (1) and (2) is a QPP.

The problem of inventory planning

Suppose a company is planning how much to produce in each quarter of a year. Let the estimated sales per quarter are s_1,s_2, s_3 and s_4. At the beginning of the year, inventory $y_0 = 0$, inventory $y_j (j = 1, \dots, 4)$ at the end of each quarter is required to be non-negative and no delay in meeting demand being admissible. Let $x_j (j = 1, \dots, 4)$ denote production in j^{th} quarter. Then we have

$$x_1 - y_1 = s_1 \qquad = b_1, \text{say},$$

$$x_1 + x_2 - y_2 = s_1 + s_2 \qquad = b_2, \text{say}, \tag{6}$$

$$x_1 + x_2 + x_3 - y_3 = s_1 + s_2 + s_3 \qquad = b_3, \text{say},$$

and $x_1 + x_2 + x_3 + x_4 - y_4 = s_1 + s_2 + s_3 + s_4 \quad = b_4, \text{say},$

where x_j and $y_j \geq 0$. (7)

Let the cost of production and storage in j^{th} quarter are $\alpha {x_j}^2$ and βy_{j-1} respectively. The total cost of storage and production C is given by

$$C = \alpha\,({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2) + \beta\,(y_1 + y_2 + y_3). \tag{8}$$

The problem now is to determine production x_j and inventory y_j ($j = 1, \ldots, u$) such that the requirements in (6) and (7) are meet and at the same time the total cost of production and storage is minimized. Clearly this is a problem of QPP.

The blending problem

Let n products are to be made by blending m types of raw materials which are available in limited supplies b_1, b_2 ,..., b_m units per period. Let the specific gravities of the blended products be required as $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively while the specific gravities of the raw materials are $\beta_1, \beta_2, \ldots, \beta_k$. The gravity specification of the j^{th} product will then be

$$\frac{\beta_1 x_{1j} + \beta_2 x_{2j} + \cdots + \beta_m x_{mj}}{x_{1j} + x_{2j} + \cdots + x_{mj}} = \alpha_j$$

or $$\frac{\sum_{i=1}^{m} \beta_i x_{ij}}{\sum_{i=1}^{m} x_{ij}} = \alpha_j, \quad j = 1, \ldots, n. \tag{9}$$

where x_{ij} is the quantity of i^{th} raw material in the j^{th} product, measured in units of volume. Requirement (9) can be expressed as a linear equation:

$$(\beta_1 - \alpha_j)x_{1j} + (\beta_2 - \alpha_j)x_{2j} + \cdots + (\beta_m - \alpha_j)x_{mj} = 0$$

or $$\sum_{i=1}^{m} (\beta_i - \alpha_j)x_{ij} = 0, \; j = 1, \ldots, n \tag{10}$$

The following restrictions on the supplies of raw materials are:

$$\sum_{j=1}^{n} x_{ij} \le b_i, i = 1, \ldots, m. \tag{11}$$

Obviously we must also have

$$x_{ij} \ge 0,\ \ j = 1, \ldots, n,\ \ i = 1, \ldots, m \tag{12}$$

Let the purchase prices q_i (i=1,..., m) of the raw materials are constant and the selling prices p_j (j=1,..., n) of the products depend linearly on the quantities produced and sold x_j (j=1,..., n), that is

$$P_j = u_j - v_j x_j,\ j = 1, \ldots, n$$

where $x_j = \sum_{i=1}^{m} x_{ij}$, $j = 1, \ldots, n$, u_j and v_j are known constants.

The total profit P per period will thus be

$$P = \sum_{j=1}^{n}\left(u_j - v_j x_j\right) x_j - \sum_{i=1}^{m} q_i \left(\sum_{j=1}^{n} x_{ij}\right). \tag{13}$$

The maximization of (13) subject to (10), (11) and (12) is a QPP in x_{ij}.

The problem of stratification in multivariate sample

Ahsan and Khan (1982a) formulated the problem of determining the stratum boundaries as a problem of nonlinear programming. The problem is that of choosing the strata boundaries so that the stratified sample thus chosen gives the maximum precision for the desired estimates. In practice this is done by choosing the boundaries for an auxiliary variable which is closely related with the estimation variables. The strata boundaries obtained by the help of the given auxiliary variable may produce better results for some of the estimation variables while worst for the others. In such cases a strategy would be to put some lower limits upon the precisions of less important variables and maximize the precision for the most important one.

The situation involving several estimation variables each having joint lognormal distribution with the stratification variable is considered and the problem is formulated as a nonlinear programming problem. It was found that the functions in the above problem are so involved that it is hard even to test them for convexity and much effort is required in obtaining the absolute minimum by using the existing non-linear programming techniques.

The above discussed problem may be solved with some ease by approximating its objective function by a quadratic function. The procedure used is that of 'Convex Chebyshev Approximation' given by Sukhovisky and Advdeyeva (1966), which works well if the function to be approximated, is smooth. If the approximated quadratic function turns out to be convex and the constraints of the problem are linear functions then we can have an approximate solution to the nonlinear programming problem by solving a quadratic programme. The computational experience suggests that a suitable choice of the starting point in the procedure may produce the desired convexity (or concavity) properties in the approximated quadratic function. Further, if the constraints of the problem are also nonlinear then they can be linearised by using the method derived by Miller (1963).

The problem of optimum allocation in multivariate stratified sampling

Ahsan and Khan (1982b) also formulated the allocation problem in multivariate stratified random sampling as a problem of nonlinear programming in which the constraint are linear. The above allocation problem can also be approximated to a QPP after approximating the objective function by a quadratic function through the convex Chebysheves approximation technique used in Ahsan (1982a).

Some Other Important Applications of Quadratic Programming

Here we discuss some important practical applications of QP models in different areas.

Finance: Analysis using QP models is an established part of selecting optimum investment strategies. Perhaps (Markowitz 1959) is the first published book in this area. With x as the vector of stock investments, the Markovitz model employs the variation in return as measured by the quadratic function $\underline{x}^T D \underline{x}$, where D is the variance/covariance matrix of returns for measuring the risk. This risk is the objective function to be minimized. Constraints in the model guarantee conservation on the flow of funds and a lower bound on the expected returns from the portfolio. There may also be bounds placed on the investments in particular sectors of the economy (such as pharmaceuticals, utilities, etc.) to make sure that the model does not put too many eggs in any one basket, thus achieving diversification. Many other practical aspects of investing can easily be included by either adding appropriate constraints or modifying the objective function by including quadratic penalty terms.

For selecting the best investment strategy, several publications measure risk by different objective functions (see Murty 2008a, b). Many authors (e.g., Crum and Nye 1981; Mulvey 1987) have designed similar multiperiod quadratic generalized network flow models in which interest, dividends, and loans are modeled by means of arc multipliers.

Taxation: QP models play a very important role these days in the analysis of tax policies. Political leaders at the national and state levels are relying more and more on such analyses to forecast growth rates in tax revenues and to set various taxes at levels that are likely to ensure growth at desired rates. White (1983) gives a detailed description of such an analysis carried out for the state of Georgia.

National and state government taxes, such as sales tax, motor fuels tax, alcoholic beverages tax, personal income tax, etc., are all set at levels to ensure a healthy economic growth. Government finance is based on the assumption of predictable and steady growth of each tax over time.

If s is the tax rate for a particular tax and S_t the expected tax revenue for this tax in year t, then a typical regression equation used to predict S_t as a function of s and t is $\log_e S_t = a + b_t + cs$ where a, b, c are parameters to be estimated from past data to give the closest fit by the least squares method, a QP technique. The annual growth rate in this tax revenue is then the regression coefficient b multiplied by 100 to convert it to percent.

The decision variables in the model are s_j = the tax rate for tax j in the base year (0^{th} year) as a fraction. From the known tax base for tax j in the 0^{th} year, the revenues from tax j in this year can be obtained as s_j (tax base for tax j)= x_j. The instability or variability in this revenue is measured by the quadratic function $(\underline{x}) = \underline{x}^T V \underline{x}$, where V is the variance/covariance matrix estimated from past data. $Q(\underline{x})$ is to be minimized. The constraints in the model consist of bounds on the x_j and a condition that $\sum x_j = T$, the total expected tax revenue in the 0^{th} year. And there is an equation that the overall growth rate which can be measured by the weighted average of the growth rates of the various taxes j, $\sum(x_j b_j)/T$ should be equal to the desired growth rate λ. Any other linear constraints that the decision variables are required to satisfy can also be included. In fact, λ can be treated as a parameter and the whole model solved as a parametric QP model. Exploring the optimum solution for different values of λ in the reasonable range yields information for the political decision makers to determine good values for the various tax rates that are consistent with expected growth in tax revenues.

Equilibrium Models: Economists use equilibrium models to analyze expected changes in economic conditions, predict prices, inflation rates, etc. These models often involve QPs. As an example, in (Glassey 1978), a simple equilibrium model of interregional trade in a single commodity is described. He considers N regions and the following data elements and variables.

Data: $a_i > 0$ the equilibrium price in the ith region in the absence of imports and exports.

$b_i > 0$ the elasticity of supply and demand in the i^{th} region.

c_{ij} the cost/unit to ship from i to j.

Variables: p_i equilibrium price in the i^{th} region.

y_i net imports into the i^{th} region (may be > 0, or 0, or < 0)

x_{ij} actual exports from region i to region j.

If $p_i > a_i$, supply locally exceeds demand in the i^{th} region, the difference being available for export. From this we have $p_i = a_i - b_i y_i$. Also, the y_i and x_{ij} are linked through flow conservation equations. The interregional trade equilibrium conditions are

$p_i + c_{ij} \geq p_j$ for all i, j

$(p_i + c_{ij} - p_j)\, x_{ij} = 0$ for all i, j

If the first condition above does not hold, exports from i to j will increase until the elasticity effects in markets i and j rise, and prices will adjust so that additional profit from export no longer exists. Also, if $x_{ij} > 0$, we must have $p_i + c_{ij} - p_i = 0$.

It can be verified that these conditions are the first-order necessary optimality conditions for a quadratic network flow problem in which the quadratic objective function can be interpreted as a net social payoff function. Using this observation (Glassey 1978) describes a procedure for computing the equilibrium prices and flows based on solving the QP.

In the same way traffic engineers use traffic equilibrium models solved by quadratic network flow algorithms for road and communication network planning. These traffic equilibrium models typically have hundreds of thousands of variables and constraints and are probably the largest QP models solved on a regular basis.

Electrical Networks: Even during the physicist J.C.Maxwell's time in the second half of the 19th century, it has been well recognized that the equilibrium conditions of an electrical or a hydraulic network are attained at the point where the total energy loss is minimized. Dennis (1959) has formally shown that the sum of the energy losses in the resistors and at the voltage sources in an electrical network, is a quadratic function of the branch currents, if all devices in the network are of a linear (i.e., ohmic) nature. Using this he formulated the problem of determining the branch currents at equilibrium in an electrical network connecting various devices, voltage sources, diodes, and resistors, as a QP. He then showed that the optimality conditions for this QP are precisely the Kirchoff laws governing the equilibrium conditions of the network, with the Lagrange multipliers representing node potentials. In the distribution of electrical power, this QP model is used to solve the load flow problem concerned with the flow of power through the transmission network to meet a given demand.

Power System Scheduling Problem: The economic dispatch problem in an electrical power system operation deals with the problem of allocating the demand for power - or system load - among the generating units in operation at any point of time. The optimal allocation of load among the units to achieve a least cost allocation depends on the relative efficiencies of the units and can be modeled as a QP (see Wood 1984). In power system operation, this model is usually solved many times during the day with appropriate load adjustments.

Application in Solving General Nonlinear Programs: At the moment, one of the most popular algorithms for solving general nonlinear programming problems is the SQP (sequential or recursive quadratic programming) method. It is an iterative method that in each iteration solves a convex QP to find a search direction and a line search problem (1-dimensional minimization problem for a merit function) in that direction. The original concepts of this method are outlined in (Wilson 1963; Han 1976; Powell 1978), but it has been developed into a successful approach through the work of many researchers (see Eldersveld 1991; Bazaraa et al. 2006 of Chap. 5; Murty 1988 of Chap. 2). The success of these methods has made QP a very important topic in mathematical programming. A nice software package for nonlinear programs based on this approach is FSQP (Zhou and Tits 1992).

Chapter - Five

Quadratic Bilevel Programming Problem

Bilevel Programming Problem

Bilevel programming problems have been introduced to the optimization community in the seventies of the 20^{th} century, although its first formulation dates back to 1934 when it has been formulated by H.V. Stackelberg in a monograph on market economy. The original formulation for the bilevel programming appeared in a paper authored by Bracken and McGill (1973), although it was Candler and Norton (1977) that first used the terms bilevel & multilevel programming. Motivated by the game theory of Stackelberg, several authors studied bilevel programming intensively and contributed to its proliferation in the mathematical programming community. A sequential optimization problem in which independent decision makers act in a noncooperative manner to maximize their individual benefits may be categorized as a Stackelberg game. The Stackelberg game is conceptually extended to the multilevel programming problem, in which the players are required to move in turn & the strategy sets are no longer assumed to be disjointed. Here decision problems involving multiple agents invariably lead to conflict & gaming. Multi-agent systems have been analyzed using approaches that explicitly assign to each agent a unique objective function and set of decision variables. The system is defined by a set of different constraints for each agent. The decisions made by each agent in these approaches affect the decisions made by the others and their objectives. There is a hierarchical ordering of agents and one set has the authority to strongly influence the preferences of the other agents. The final decisions are executed sequentially within the hierarchy, from highest to lowest levels. Bilevel Programming problem is a special case of multilevel programming problems with a structure of two levels, namely, the upper level and the lower level. The upper level decision maker is called the leader and that of the lower level is called the follower. The follower executes its policies after & in view of, the decisions of upper level decision maker. Control over the decision variables is partitioned among the levels, but a decision variable of one level may affect the objective function of other level. Thus, an important feature of bilevel programming problems is that a planner at one level of the hierarchy may have his objective function and decision space determined, in part by variables controlled at other level. However, his control instruments may allow him to influence the policies at other level and thereby improve his own objective function.

The problems we want to consider have the following common characteristics.

1) The system has interacting decision-making units within a hierarchical structure.
2) The execution of decision is sequential from upper to lower level. The follower executes its policies after, and in view of, the decisions of the leader.
3) Each decision-making unit optimizes its own objective function independently of other units, but is affected by the actions and reactions of the other unit.
4) The external effect on a decision maker's problem can be reflected in both his objective function and his set of feasible decisions.

Let us consider a bilevel hierarchical system where a vector of decision variables $(x,y) \in \Re^n$ be partitioned among the two, upper and lower level decision makers i.e. leader and follower respectively. The leader has the control over the decision variable $x \in \Re^{n_1}$ and follower over the variable $y \in \Re^{n_2}$, where $n_1 + n_2 = n$. Furthermore, assuming that $F, f: \Re^{n_1} \times \Re^{n_2} \to \Re^1$

are linear and bounded, the linear bilevel programming problem can be stated as follows:

P1: $max_x\, F(x,y) = ax + by\ where\ y\ solves$

P2: $max_y\, f(x,y) = cx + dy$

$s.t.\, Ax + By \leq r$

where $a, c \in \Re^{n_1}$, $b, d \in \Re^{n_2}$, $r \in \Re^m$, A is an $m \times n_1$ matrix, B is an $m \times n_2$ matrix. Let the feasible choices of (x, y) be denoted by the constraint region $S = \{(x,y)|\mathrm{Ax} + \mathrm{By} \leq \mathrm{r}\}$. Hence for each value of x, lower level will react with a corresponding value of y. This induces a functional relationship between the decisions of leader and the reactions of the follower. For a given x, let $Y(x)$ denote the set of optimal solutions to the inner problem, P2,

$$\max_{y \in Q(x)} \tilde{f}\,(y) = dy \text{ where } Q(x) = \{y|By \leq r - Ax\}$$

and represent the upper level decision maker's solution space, or the set of rational reactions of f over S, as

$$\Psi_f(S) = \{(x,y) \in \mathrm{S}, \mathrm{y} \in \mathrm{y(x)}\}.$$

We assume that S and $Q(x)$ are bounded and non-empty. The definitions of feasibility and optimality for the linear bilevel programming problem are given by the following:

Definition 1. A point (x,y) is called feasible if $(x, y) \in \Psi_f(S)$.

Definition 2. A feasible point (x^*, y^*) is called optimal if $ax^* + by^*$ is unique for all $y^* \in Y(x^*)$, and $ax^* + by^* \geq ax + by$ of all feasible pairs $(x,y) \in \Psi_f(S)$.

Algorithms for Bilevel Programming Problem

Many researchers have designed algorithm for the solution of the bilevel programming problem. One class of techniques consists of extreme point algorithms that have been mostly applied to the linear bilevel programming problem. Another class of algorithms uses Kuhn-Tucker approach that replaces the linear bilevel programming problem by its Kuhn-Tucker conditions and solves a set of equalities and inequalities. On the other hand, decent algorithms constitute an important class of algorithms for nonlinear bilevel programming problem. However, it is assumed for almost all those decent algorithms that the solution set of the lower level problem is a singleton for any given value of the upper level variables. Under this assumption bilevel programming problem can be transformed into a single level optimization problem where the lower level variables are taken as a function of the upper level variables. On the basis of the gradient information generated from the lower level optimization problem, Kolstad and Lasdon (1990) proposed a heuristic descent algorithm for the bilevel programming problem. Vincete et al. (1994) and Jiye et al. (2000) presented a descent method for solving quadratic bilevel programming problem.

Here we discuss in brief the above mentioned two approaches for linear bilevel programming problem i.e. vertex enumeration approach and Kuhn-Tucker approach.

Vertex Enumeration Approach

The first method using such an approach was proposed by Candler and Townsley (1982). They observed that once an optimal basis to the inner problem was obtained, changing x might affect its feasibility, but not its optimality. Thus, they proposed a scheme that involved implicit enumeration of adjacent bases to test for feasibility and optimality. Bialas and Karwan (1984) developed a similar vertex enumeration procedure called the Kth-best algorithm. In their approach, the leader solves his problem with respect to both leader and follower decision vectors, and would order all the basic feasible solutions in such a way that

$$F(x^k, y^k) \geq F(x^{k+1}, y^{k+1}), \qquad k = 1,2 \ldots \ldots$$

At any iteration k, given x^k, the follower will solve his problem to obtain a solution y^{*k}. If $y^{*k} \neq y^k$, then the algorithm proceeds to the next best solution for the leader (x^{k+1}, y^{k+1}) and the follower's computation is repeated. Optimality is reached when $y^{*k} = y^k$.

Computational experience with the *Kth-best* algorithm has demonstrated that it finds a solution easily for most problems, although occasionally an unacceptable long time may be needed before a solution is found.

Narula and Nwosu (1983, 1985) also proposed a procedure via regular simplex pivots with modification after taking the dual of the problem P2. Shi et al. (2005a) extended *Kth-best* approach and also applied it for linear bilevel multi-follower programming problem (Shi et al. 2005b).

Kuhn-Tucker Approach

In Kuhn-Tucker approach the rational reaction set of the follower is replaced by Kuhn-Tucker optimality conditions. The leader takes into account the follower's optimality conditions while solving its own problem. Thus, taking the Kuhn-Tucker transformation to the inner problem P2, the resulting equivalent problem to P1 can be written as

P3: $$\max_{x,y,w,u} ax + by$$

$$s.t. \quad wB = d$$

$$wu = 0$$

$$Ax + By + u = r$$

$$u, w \geq 0, \quad \text{where } u, w \in \mathfrak{R}^{m}$$

It has been shown by Wen (1981) and Bard (1983) that $(x, y) \in \Psi_f(S)$ in P1, if and only if there exists u, w, such that (x, y, u, w) is feasible in P3.

Attempts at solving problem P3 resulting from the Kuhn-Tucker approach include mixed integer programming by Fortuny-Amat and McCarl (1981), branch and bound technique by Bard and Moore (1990) and parametric complementary pivoting by Bialas and Karwan (1984). The parametric complementary pivoting algorithm can be viewed as an implicit enumeration of the lower level optimal bases. All of the computations for the parametric complementary pivoting algorithm may be performed within the framework of a simplex like tableau whose size is roughly that of the original system. In addition, this method can be extended to solve a three level linear problem.

The Mixed Integer Concave Quadratic Bilevel Programming Problem

Introduction: Many decision making problems in real applications naturally result in optimization formulations in a form of bilevel nonlinear programming. Several algorithms have been proposed for nonlinear bilevel programming under various assumptions. Branch and Bound approach has been extended to cover the situations where the lower level objective is quadratic and the lower level constraints are linear. In this case, one replaces the lower level problem by the equivalent Kuhn-Tucker system, whose equalities and inequalities are linear, with the exception of the complementarily constraint. The latter being a difficult type of constraint to deal with, branch and bound methods may be used to defer the introduction of such constraints in the solution process for as long as possible. This is the basic idea underlying the approaches of Edmunds and Bard (1991), Al-Khayyal et al. (1992) and more recently Thoai et al. (2002). Some authors have proposed descent methods for solving bilevel nonlinear programs. However, it is assumed for almost all these descent algorithms that the solution set of the lower level problem is a singleton for any given value of the upper level variables. Under this assumption, a bilevel programming problem can be transformed into a single level optimization problem where the lower level variables are taken as a function of the upper level variables. These descent algorithms heavily depend on the information about this implicit function. The computational results showed that the heuristic algorithms are quite efficient in computing an approximate solution of nonlinear bilevel programming problems. Penalty methods constitute another category of algorithms for bilevel nonlinear programming problems.

Mixed integer and integer nonlinear bilevel programming problems have received less attention in the literature. In this chapter we propose an algorithm for a class of mixed integer bilevel programming problems where the leader controls a set of continuous and discrete variables and tries to minimize a concave quadratic objective function. The follower's objective function is assumed to be linear in continuous decision space. Also the constraints are assumed to be linear. Here we also make use of Tui's cut (1964) that eliminate from feasible region only that portion which does not contain any solution better than the current local minimum.

The Algorithm

In this chapter, we consider the following concave quadratic bilevel programming problem with leader having continuous as well as discrete decision variables.

$$\min_x F(x,y) = c^T \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T D \begin{bmatrix} x \\ y \end{bmatrix}$$

where y solves

$$\min_y f(x,y) = d_1x + d_2y$$

$$s.t.\ Ax + By \leq b$$

$$x, y \geq 0$$

$$x_{i+1}, \dots, \mathrm{x_n} \in \mathrm{Z^+}$$

where $x \in \Re^{n_1}$ is a vector of leader's problem variables, of which i are continuous and $(n_1 - i)$ are integers.

$$\mathrm{d_1} \in \Re^{n_1}, \mathrm{d_2}, \mathrm{y} \in \Re^{n_2}, c^T \in \Re^{n_1+n_2}, \mathrm{A} \in \Re^{p\times n_1}, \mathrm{B} \in \Re^{p\times n_2}, \mathrm{b} \in \Re^{p}$$

D is a symmetric and negative semi definite matrix of order $(n_1 + n_2)$.

The leader's problem can be written as

$$\min_x F(x,y) = c^T \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T D \begin{bmatrix} x \\ y \end{bmatrix}$$

$$s.t.\ Ax + By \leq b \qquad (1)$$

$$x, y \geq 0$$

$$x_{i+1}, \dots, \mathrm{x_n} \in \mathrm{Z^+}$$

Since $F(x,y)$ is concave, it is known that the global minimum for leader's problem lies on an extreme point of the set

$$S_1 = \{(x,y) | Ax + By \leq b\}$$

Let (x^m, y^m) be the extreme point of S_1. We construct a linear function for $F(x,y)$ as

$$L_m(x,y) = \nabla F(x,y)^T (x,y)$$

From concavity of $F(x,y)$ it follows that

$$L_m(x,y) \geq F(x,y) \qquad \forall (x,y)$$

Now suppose that (x^{m_1}, y^{m_1}) is a local solution to the leader's problem (1). We solve the following auxiliary problem

$$\min_x L_{m_1}(x,y) = \nabla F(x^{m_1}, y^{m_1})^T (x,y) \qquad (2)$$

$$s.t. \quad (x,y) \in S_2$$

where $S_2 = S_1$, except that if we introduce into the problem a Tui's cut S_2 will be the intersection of S_1 with this cut.

Let (x^{L_1}, y^{L_1}) be the solution to the linear programming in (2). Clearly (x^{L_1}, y^{L_1}) is an upper bound to the optimal solution of the original leader's problem.

From (x^{L_1}, y^{L_1}) we move along the various binding edges of S_1 in the search of another local minimum of the problem (1). This search is made by moving to the various adjacent extreme points until an extreme point (x^{m_2}, y^{m_2}) is reached such that there is no adjacent extreme point with a value of $F(x,y)$ smaller than $F(x^{m_2}, y^{m_2})$. The procedure is now repeated with the following auxiliary problem

$$\min_x L_{m_2}(x,y) = \nabla F(x^{m_2}, y^{m_2})^T (x,y)$$

$$s.t. \quad (x,y) \in S_1$$

Now if the $(i+1)^{\text{th}}$ auxiliary problem

$$\operatorname{Min}_x L_{m_{(i+1)}}(x,y) = \nabla F(x^{m_{(i+1)}}, y^{m_{(i+1)}})^T (x,y)$$

$$s.t. \quad (x,y) \in S_1 \qquad (3)$$

does not have a solution. The solution of the leader's problem is given by (x^{m_i}, y^{m_i}).

If the required variables from x^{m_i} are integer then move on to solve the original problem (1). Otherwise we add the Gomory's (1960b) mixed integer cut at that point to get the required variables as integers. Let the mixed integer solution be (x^0, y^0). Now for the follower's problem we fix the x vector and solve the following problem

$$\min_y f(x, y) = d_1 x + d_2 y$$

$$s.t.\ Ax + By \leq b$$

$$x = x^0$$

$$y \geq 0$$

Suppose the solution of this problem is $(x^0, \overline{y})$. If $\overline{y} = y^0$ then, (x^0, y^0) is the solution for the original problem.

The procedure can be summarized in the following steps.

Step 1.Find the local solution to the problem (1). Let it be (x^{m_1}, y^{m_1}). Construct a linear function for the concave quadratic objective function of the leader.

Step 2.Solve the following auxiliary problem

$$\min_x L_{m_1}(x, y) = \nabla F(x^{m_1}, y^{m_1})^T (x, y)$$

$$s.t. \quad (x, y) \in S_2$$

where S_2 is the new feasible region after the introduction of Tui's cut. Let the solution be (x^{L_1}, y^{L_1}).

Step 3.Now from (x^{L_1}, y^{L_1}) move along the various adjacent extreme points unless (x^{m_2}, y^{m_2}) is reached such that there is no point with a value of $F(x, y) \leq F(x^{m_2}, y^{m_2})$. Go to step 1.

Step 4.If at $(i+1)^{\text{th}}$ auxiliary problem at step 2 there is no solution, then the solution to the leader's problem (1) is given by (x^{m_i}, y^{m_i}).

Step 5.If the required variables in x^{m_i} are integers then go to 6, otherwise add Gomory's mixed-integer cut to get those variables as integers. Let this solution be (x^0, y^0).

Step 6. For a fixed $x = x^0$ solve the follower's problem. Let its optimal solution be $(x^0, \overline{y})$.

Step 7. If $\overline{y} = y^0$ then, (x^0, y^0) is the solution for the given concave quadratic bilevel programming problem. If $\overline{y} \neq y^0$ then from step 3 find the second best solution for (1), then go to step 5.

Numerical Illustration

The following example will illustrate the computational details.

$$\min_x F(x,y) = 2x + 3y - x^2 - y^2$$

where y solves

$$\min_y f(x,y) = 2x - 5y$$

$$\begin{aligned} s.t. -5x + 2y &\leq 1 \\ -x + 4y &\leq 2 \\ 2x + y &\leq 9 \\ 3x - 5y &\leq 3 \\ x &\geq 0, \text{ integer} \\ y &\geq 0 \end{aligned} \tag{4}$$

First we consider the leader's problem

$$\min_x F(x,y) = 2x + 3y - x^2 - y^2$$

$$\begin{aligned} s.t. -5x + 2y &\leq 1 \\ -x + 4y &\leq 2 \\ 2x + y &\leq 9 \\ 3x - 5y &\leq 3 \\ x &\geq 0, \text{ integer} \end{aligned} \tag{5}$$

$$y \geq 0$$

The value of the objective function at a basic feasible solution $(x_0, y_0) = (0,0)$ is

$$F(x_0, y_0) = (0,0)$$

The two extreme points adjacent to (x_0, y_0) are (0, 0.5) and (1, 0). The local minimum (x^{m_1}, y^{m_1}) will be at (x_0, y_0) since the value of objective function at both these extreme points is found to be greater than $F(x_0, y_0)$.

Thus $(x^{m_1}, y^{m_1}) = (x_0, y_0)$.

For obtaining Tui's cut the following two one variable optimization problems are solved

$$\max\ x$$

$$s.t.\ 2x - x^2 \geq 0, x \geq 0$$

and

$$\max\ y$$

$$s.t.\ 3y - y^2 \geq 0, y \geq 0$$

which yield their respective solution as x =2, y =3.

Thus Tui's cut is

$$\frac{x}{2} + \frac{y}{3} \geq 1 \tag{6}$$

The auxiliary linear programming problem for this objective function is

$$L_{m_1}(x, y) = \nabla F(x^{m_1}, y^{m_1})^T (x, y)$$

$$= 2x + 3y \tag{7}$$

Adding Tui's cut as an additional constraint in the constraint set of (5) without the integer restriction on x we get the following linear programming problem.

$$minL_{m_1}(x, y) = 2x + 3y$$

$$s.t.\ (5) \tag{8}$$

and $3x + 2y \geq 6$

The solution of (8) is

$$(x^{L_1}, y^{L_1}) = (1.71, 0.42)$$

From (x^{L_1}, y^{L_1}) we move successively to the adjacent extreme points $x_1^{L_1}, x_2^{L_1}, x_3^{L_1}$ and found that the auxiliary linear programming problem with new Tui's cut added in the constraints set is infeasible. Thus the optimal solution to the problem (5) is, x =1.71.

As is non integer. Therefore, we add Gomory's mixed integer cut in the problem (8) and obtain the solution as (2, 0.6).

Now for a given x =2 solve the follower's problem

$$\min_y f(x,y) = 2x - 5y$$

$$\begin{aligned} s.t. -5x + 2y &\leq 1 \\ -x + 4y &\leq 2 \\ 2x + y &\leq 9 \\ 3x - 5y &\leq 3 \\ x &\geq 2, \text{ integer} \\ y &\geq 0 \end{aligned} \tag{9}$$

We get the solution as $(x^*, y^*) = (2, 0.6)$.

Since $y^* = y^0$.

Therefore, this satisfies the condition for BLPP. Hence the optimal solution of concave quadratic BLPP is x =2, y =0.6.

REFERENCES

Agrawal (1974a): "On mixed integer quadratic programs. *Nav. Res. Log. Quart.* Vol.21, 289-297."

Agrawal (1974b): "On integer solution to quadratic programs by a branch and bound technique". *Trabajos de Estatistica Y De Investigation Operativa*, Vol.25, 65-70.

Ahsan and Khan (1982a): "Minimizing a Non Linear Function Arising in Stratification through Approximation by a Quadratic Function". Section of Statistics. Proceeding of the 70th session of *Indian Science Congress Association.*

Ahsan and Khan (1982b): "Optimal Allocation in Multivariate Strarified Random Sampling with Overhead Cost". *Metrika* , Vol.29, 71-78.

Al-Khayyal et al. (1992): Global Optimization of Concave Funtions subject to Quadratic Constraints: An application in Nonlinear Bilevel Programming. *Annals of Operations Research*, 34, 125-147.

Arrow, K. J., L. Hurwicz and H. Uzawa (1958): Studies in Linear and Non-Linear Programming, Standford Mathematical Studies.

Arshad, Khan and Ahsan (1981): "A procedure for solving concave quadratic programs". *Aligarh Journal of Statistics*, Vol.1, No.2, 106-112.

Bala's (1969): "Duality in Discrete Programming: II. The quadratic case". *Management Science* 16, 14-32.

Bard (1983): An Efficient Point Algorithm for a Linear Two-Stage Optimization

Problem. *Operations Research*, 31, 670-684.

Bard and Moore (1990): The Mixed Integer Linear Bilevel Programming Problem. *Operations Research*, 38, 911-921.

Bari and Arshad (1978): "A branch and bound method for integer quadratic programming". *Pure and Applied Mathematika Science*, Vol. 7, 43-46.

Bazaraa M.S., Jarvis John J, Sherali H D: Linear Programming and Network Flows, 2nd edition.

Bazara, M.S., Sherali, D. and Shetty, C.M. (2004): Non Linear Programming: Theory and Algorithm, Second Edition.

Beale (1959): "On Quadratic Programming". *Nav. Res. Log. Quart.* Vol.6, 227-243.

Beale (1967): "Numerical Methods" in "Non-Linear Programming", by *J. Abadie, Norh Holand Publishing Co., Amsterdam*, 135-204.

Bialas and Karwan (1984): Two level Linear Programming. *Management Science*. 30(8), 1004-1020.

Bracken and McGill (1973): Mathematical Programs with Optimization Problems in the constraints. *Operations Research*, 21, 37-44.

Brankin and Dorfman (1956): "A method for Quadratic Programming" *Econ.*, 42.

Candler and Norton (1977): Multilevel Programming. *Tech. Rep.*20, *World Bank Development Research Center, Washington D.C.*

Candler and Townsley (1982): A Linear Two-Level Programming Problem. *Comput. Oper. Res.*9, 59-76.

Charnes and Cooper (1961): "Management models and industrial operations of linear programming". *John Wiley and Sons. Inc., New York.*

Cottle and Mylander (1970): "Ritter's cutting plane method for non- convex quadratic programming" in "Integer and Non-Linear Programming" Edited by *J.Abadie, North Holland Publishing Co., Amsterdam.*

Crum RL and Nye DL (1981): A network model of insurance company cash management. *Math Program Stud* 15: 86-101.

Dennis JB (1959): Mathematical Programming and electrical networks. *Wiley, NY.*

Edmunds and Bard (1991): Algorithms for Nonlinear Bilevel Mathematical Programs. *IEEE Transactions on Systems, Man and Cybernatics*, 21(1), 83-89.

Eldersreld SK (1991): Large scale sequential quadratic programming, SOL91. *Department of OR, Stanford University, CA.*

Fortumy-Amat and McCarl (1981): The Potential Role of Multilevel Programming in Agricultural Economics. *Am. J. Agric. Econ.*, 63, 521-531.

Frank and Wolfe (1956): "An Algorithm for Quadratic Programming". *Nav. Res. Log. Quart.*, 3.

Frisch(1957): "The multiplex method for Linear and Quadratic Programming". *Mem. Univ. Social Inst. of Oslo.*

Geoffrion (1972): "Generalized Bender's Decomposition". *Journal of Optimization Theory and Applications* 10, 237-260.

George B Thapa: Linear Programming, *Springer.*

Gerard Sierksma: Linear and Integer Programming.

G Hadley: Linear Programming, *Narosa Publication*, 8th edition.

Glassey CR (1978): A quadratic network optimization model for equilibrium single commodity trade flows. *Math Problem* 14: 98-107.

Gomory's (1960b): An Algorithm for Mixed-Integer Problem. RM-2597. *RAND Corp. Santa Monica, California.*

Han SP (1976): Super linearly convergent variable metric algorithms for general nonlinear programming problems. *Math Program 11*: 263-282.

Hildreth (1957): "A Quadratic Programming Procedure". *Nav. Res. Log. Quart.*, 4.

Houthakker (1960): "The Capacity Method of Quadratic Programming". *Econ.*, 28.

Ignizio James P & Kavalier Tom M: Linear Programming.

Jiye et al. (2000): A New Descent Algorithm for Solving Quadratic Bilevel Programming Problems. *Acta. Math. Appl. Sin.*, 16(3), 235-244.

Kolstad and Lasdon (1990): Derivative Evaluation and Computational Experience with Large Bilevel Mathematical Programs. *Journal of Optimization Theory & Apps*, 65, 485-499.

Kuhn and Tucker (1951): "Non-Linear Programming" in Proceedings of the second Berkely Symposium on Mathematical Statistics and Probability. J.Neyman, Editor. *Berkeley and Los Angeles, California: University of California Press*, 481-492.

Kunzi and Oettli (1963): "Integer Quadratic Programming" in "Recent

Advances in Mathematical Programming" *McGraw Hill Book Co., New York.*

Kunzi, Krelle and Oettli (1966): "Non-Linear Programming". *Blaisdell Publishing Co., London.*

Kwak N K & Schniederjans M J: Mathematical Programming.

Lazimy (1982): "Mixed Integer Quadratic Programming". *Math, Programming* 22, No.3, 332-349.

Markowitz HM (1959): Portfolio selection: efficient diversification of investments. *Wiley, NY.*

Miller (1963): "Recent Advances in Mathematical Programming". *McGraw Hill Book Co., New York.*

Mulvey JM (1987): Nonlinear network models in finance. *Adv Math Program Finan. Plann* 1: 253-271.

Murty KG (2008a,b): Forecasting for supply chain and portfolio management, Chap 3. In: Neogy SK, Bapat RB, Das AK, Parthasarathy T (eds) *Mathematical programming and game theory for decision making*, vol 1, pp 231-255. World Scientific, Singapore.

Narula, S.C. and Nwosu, A.D. (1983): Two-Level Hierarchical Programming Problem. In Essays and Surveys on Multiple Criteria Decision Making, *Lecture Notes in Economics and Mathematical Systems (P.Hansen Ed.)*, 290-299. *Springer, New York.*

Narula, S.C. and Nwosu, A.D. (1985): An algorithm to solve a Two- Level Resource Control Pre-Emptive Hierarchical Programming Problem. In Mathematics of Multiobjective Optimization (P. Serafini, Ed.), 353-373, *Springer, New York.*

Paul R Thie & G E Keough: Introduction to LP and Game Theory, 3rd edition, john wiley & sons.

Powell MJD (1978): Algorithms for nonlinear constraints that use Lagrangian functions. *Math Program*, 14:224-248.

Ritter (1965): "Stationary Points of Quadratic Maximum Problem". *Z. Wahrscheinlichkeitstheorie and Verw. Gebiete*, Vol.4, 149-158.

Ritter (1966): "A method for solving maximum problems with a non- concave quadratic objective function". *Z. Wahrscheinlichkeitstheorie and Verw. Gebiete*, Vol.4, 340-351.

Rosen (1960): "The Gradient Projection Method for Non-Linear Programming Part I, Non-Linear Constraints", *J.SIAM*, Vol.3.

Rosen (1961): "The Gradient Projection Method for Non-Linear Programming Part II, Linear Constraints", the Royal Symposium or Mathematical Progression, Proceedings of a Conference, R-351 (1960), *J.SIAM*, Vol.8, 101-217"

Shi, C., Lu,J. and Zhang, G. (2005a): An Extended Kth-Best Approach for Linear Bilevel Programming. *Applied Mathematics and computation*, 164, 843-855.

Shi, C., Zhang, G. and Lu, J. (2005b): The Kth-Best Approach for Linear Bilevel Multi-Follower Programming. *Journal of Global Optimization*, 33, 563-578.

Thoai, N.V., Yamamoto, Y. and Yashise, A. (2002): Global optimization method for solving mathematical programs with linear complementarity constraints. Discussion paper no.987, Institute of

Policy and Planning Sciences, *University of Tsukuba, Japan*, May 2002.

Tui (1964): "Concave Programming Under Linear Constraints" *Soviet Mathematics*, Vol.5, 1437-1440.

Van De Panne (1961): "Quadratic Programming as an extension of conventional quadratic maximization". *J. Inst. Manag. Sci.*, 87.

Van De Panne (1975): A Node method for Multiparametric Linear Programming, *Mang. Sci.*, Vol 21, No.9, 1975.

Vincete et al. Savard, G. and Judice, J. (1994): Descent Approaches for Quadratic Bilevel Programming. *Journal of Optimization Theory & Applications*, 81, 379-399.

Wen UP (1981): Mathematical Methods for Multilevel Linear Programming. Ph.D. *Dissertation, Department of Industrial Engineering, State University of New York at Buffalo, New York.*

White PC (1983): Trade-off in growth and stability in state Taxes. *Natl Tax J* 36:103-114.

Wilson RB (1963): A simplicial algorithm for convex programming, *Ph.D. dissertation, School of Business Administration, Harvard.*

Wolf (1959): "The Simplex Method for Quadratic Programming". *Econ.*, Vol. 27, 382-398.

Wood AJ (1984): Power generation, operation and control. *Wiley NY.*

Zhou JL and Tits AL (1992): User's guide to FSQP version 3.1. SRC TR-92-107r2, Institute for Systems Research, *University of Manyland, College Park.*

Zoutendjik (1960): "Methods of Feasible Direction". A study in Linear and Non-Linear Programming. *Amsterdam, London, New York and Princeton: Elsevier.*

Zwart (1974): "Global maximization of a convex function with linear inequality constraints". *Operat. Res.* Vol.22, 602-609.